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FINAL REPORT

Optimum Digital Filter Study for Strapdown Inertial Navigation System Initial Alignment

ORDNANCE DEPARTMENT
OF THE ELECTRONIC SYSTEMS DIVISION
GENERAL ELECTRIC
100 PLASTICS AVENUE, PITTSFIELD, MASSACHUSETTS

FINAL REPORT

Optimum Digital Filter Study for Strapdown Inertial Navigation System Initial Alignment

Prepared for

National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama

Contract No. NAS 8-21273

July 8, 1968

ORDNANCE DEPARTMENT
OF THE ELECTRONIC SYSTEMS DIVISION
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CONTENTS

I. INTRODUCTION

This report summarizes the work performed by the Ordnance Department of the General Electric Company for the George C. Marshall Space Flight Center under contract NAS-8-21273. This work was performed over the period from January 8, 1968 to July 8, 1968.

The basic objective of this study was to investigate digital methods for recovery of the initial misalignment of a strapdown inertial navigation system from vibration- and sway-corrupted data on the launch pad. The methods investigated under this contract are all based on the criterion of a minimum mean square error and differ mainly in the mechanization technique and the amount of a priori knowledge of the statistics which is assumed. Methods considered included an optimal filter of the Kalman type, a simplified Kalman filter, a Maximum Likelihood approach, and a Least Squares Curve Fit technique with and without preconditioning of the data.

Section II of this report contains a summary of the results of this study, the conclusions reached regarding recommended filtering techniques, and information to allow trade-offs between the two recommended techniques to be performed as more information is obtained about the noise characteristics and operational restrictions (time, computer capacity available, etc.)

Section III of this report contains a description of the analysis performed for the various filtering methods considered.

Section IV contains a description of the erection and alignment simulation program which was provided to the Marshall Space Flight Center to allow further evaluation of the two recommended techniques.

Section V summarizes the results of the work done in determining the computer requirements for solution of the filter and azimuth alignment equations for the two recommended filters.

Appendix A contains a discussion of the form of a strapped down accelerometer output in a swaying missile and the terms which must be filtered out of this output in order to determine the accelerometer orientation.

Appendix B contains a discussion of the optical alignment equations.

Appendix C contains a derivation of the closed form solution of the least squares curve fit filter equations.

II. RESULTS AND CONCLUSIONS

A. CONCLUSIONS

There are several filtering techniques which provide 1 σ accuracies better than 20 seconds of arc with a data-gathering period less than 100 seconds of time.

The recommended filtering technique is the optimal 4-state filter (simplified Kalman). This recommendation is based mainly on a consideration of performance. The performance of this filter is equivalent to that of the maximum likelihood filter and has less computational complexity. This filter also provides better accuracy in a given time than the least squares curve fit approach. If the effects of earth's rate coupling are removed by either updating the transformation matrix and releveling, or by using the correction matrix derived in Section III-F, the performance approaches that of the optimal 12-state filter with a significant decrease in computational complexity required.

The main disadvantages of the optimal 4-state filter relative to the least squares curve fit filter are the launch pad computer capacity and the amount of precomputation required. If pad computer capacity becomes a limiting item, the least squares filter provides a back-up technique which can be used to alleviate the problem. The linear least squares filter using non-integrated data is the recommended approach in this case.

Gyro drift causes an error which is equal to one half the drift angle accumulated over the filtering time in both the 4-state optimal filter and the least squares filter. The Marshall Space Flight Center has indicated that gyro drift will be small enough so that this error will not be significant.

Bias errors due to earth's rate crosscoupling exist in both the 4-state optimal and least squares filters. These errors can be removed by either updating the transformation matrix and releveling or by using the correction matrix derived in Section III-F. It is recommended that the correction matrix approach be taken to eliminate the need for additional iterations.

The maximum likelihood and least squares curve fit filter using integrated data are not recommended for further consideration - the former because it is extremely complex to mechanize and the later because it is extremely sensitive to sway velocity noise correlation time.

B. RESULTS SUMMARY

Figures 1 through 10 show the results of the analysis of the Kalman filter. Figure 1 is the 1 σ erection error for the optimal 12-state filter. This filter is described by equations 3 - 72 to 3 - 76. The conditions used for this baseline case are given by

Initial erection uncertainty	σ_θ	=	1/2 degree
Gyro drift uncertainty	σ_d	=	.1 meru
RMS sway velocity	σ_v	=	.5 m/s
Center frequency of sway velocity power spectrum	f	=	.25 cps
Correlation time (reciprocal of bandwidth) of sway velocity power spectrum	τ	=	200 sec
Sampling rate			1 sample/sec

It is seen that the performance of this filter is well within the required accuracy. The 1 σ error after 60 seconds of time is less than 6 arc seconds.

Figure 2 shows the performance of the suboptimal 4-state filter for various initial erection angles. This filter is described by equations 3-94 to 3-100. The main simplification employed in reducing the 12-state optimal filter to two identical 4-state filters was the neglect of earth's rate crosscoupling terms in the system dynamics. In addition, gyro drifts were neglected in the simplified dynamics. The difference in errors for various initial angles is attributable to the earth's rate coupling. For initial angles less than 0.5 degrees the increase in error over the optimal filter is less than 1 second of arc at 60 seconds time. If the initial angle uncertainty is large, the 12 state optimal filter performance can be approached by updating the computer coordinate system after about 20 seconds to reduce the crosscoupling effects. Another approach to removing the crosscoupling terms is to use the correction matrix derived in Section III-F.

Figure 3 shows the degradation in performance due to the neglect of gyro drift. It is seen that the drift uncertainty must be quite large (20 meru = 0.3 deg/hr) before the filter performance is degraded appreciably.

Figure 4 shows the erection error vs time for various sampling times. A sampling time of 1 sec seems to provide a good trade-off between computational requirements and erection accuracy. The worst case occurs when the sampling time is 4 sec. This is because the center frequency of the sway velocity power spectrum is .25 cps. Since the power spectrum is very narrow, the sway velocity is almost sinusoidal with a period of 4 sec. The sampling is thus at the same part of the sine wave and it becomes quite difficult to filter out the noise.

The correlation time (reciprocal of the bandwidth) of the narrow band noise has a significant effect on the estimation accuracy. This is demonstrated in Figure 5. It was

Figure 1. THE 1 σ ERECTION ERROR FOR THE OPTIMAL FILTER

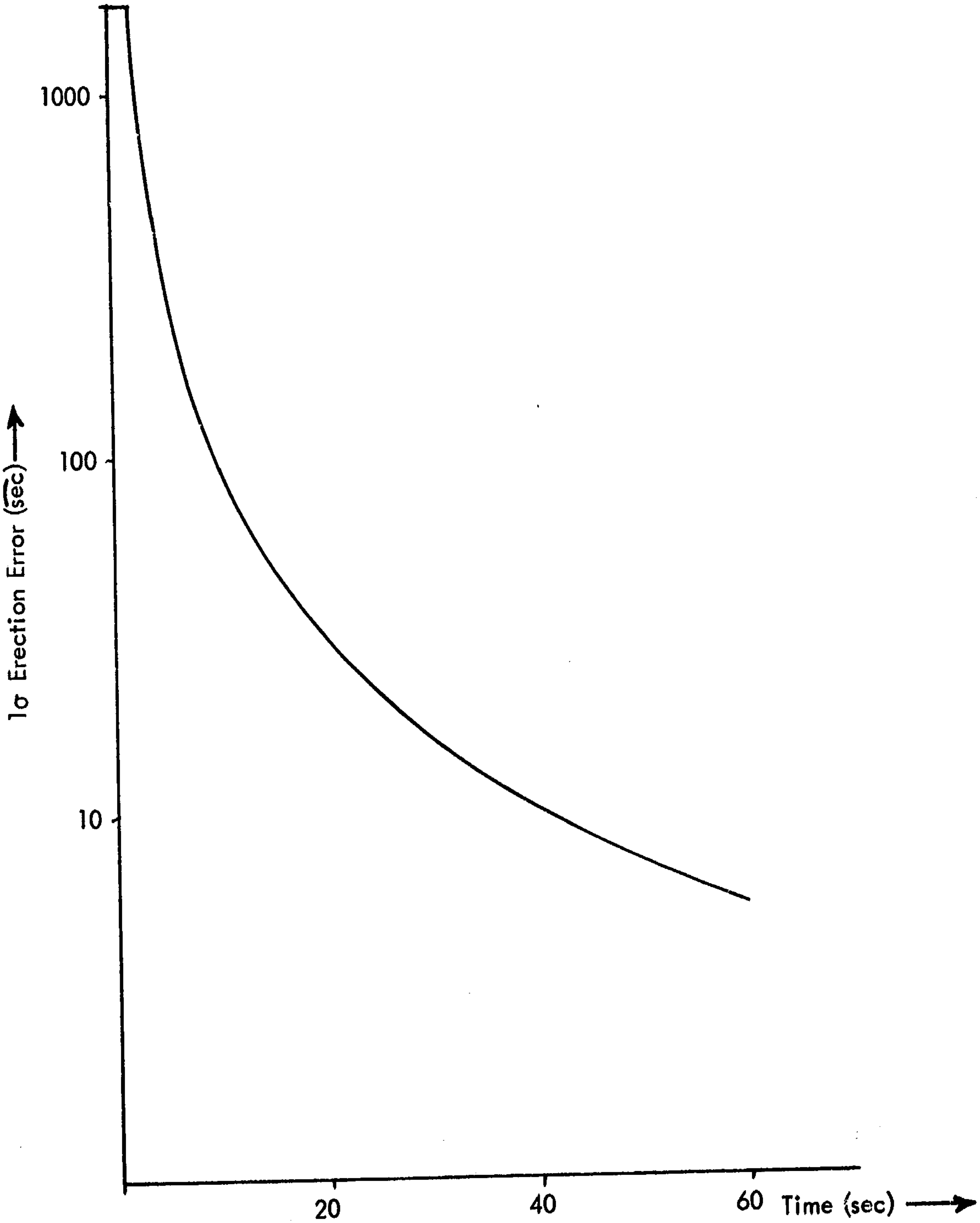
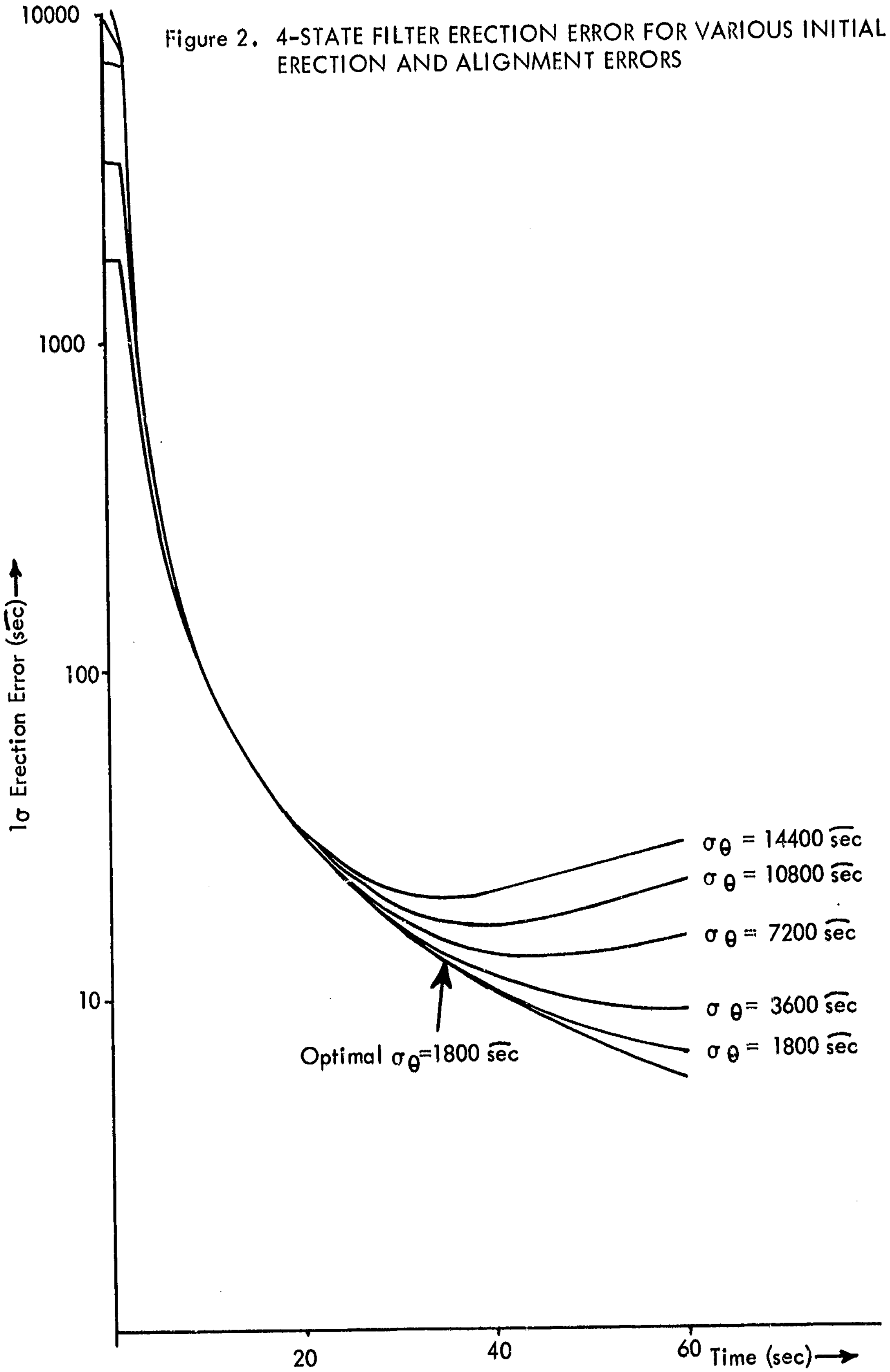


Figure 2. 4-STATE FILTER ERECTION ERROR FOR VARIOUS INITIAL ERECTION AND ALIGNMENT ERRORS



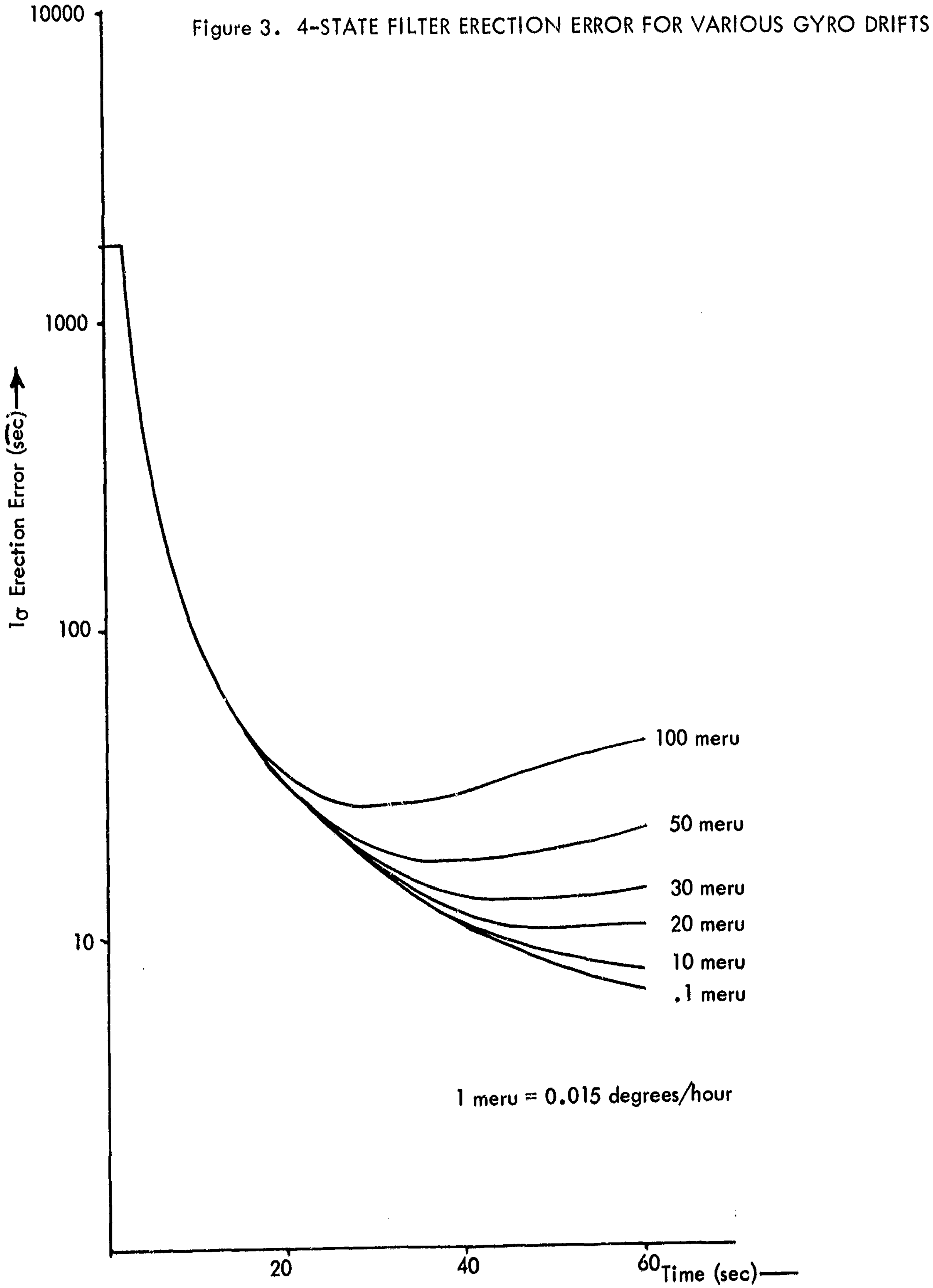


Figure 3. 4-STATE FILTER ERECTION ERROR FOR VARIOUS GYRO DRIFTS
10000
1000
100
10
1σ Erection Error (sec)
100 meru
50 meru
30 meru
20 meru
10 meru
.1 meru
1 meru = 0.015 degrees/hour
20
40
60
Time (sec)

Figure 4. ERECTION ERROR FOR VARIOUS SAMPLING TIMES

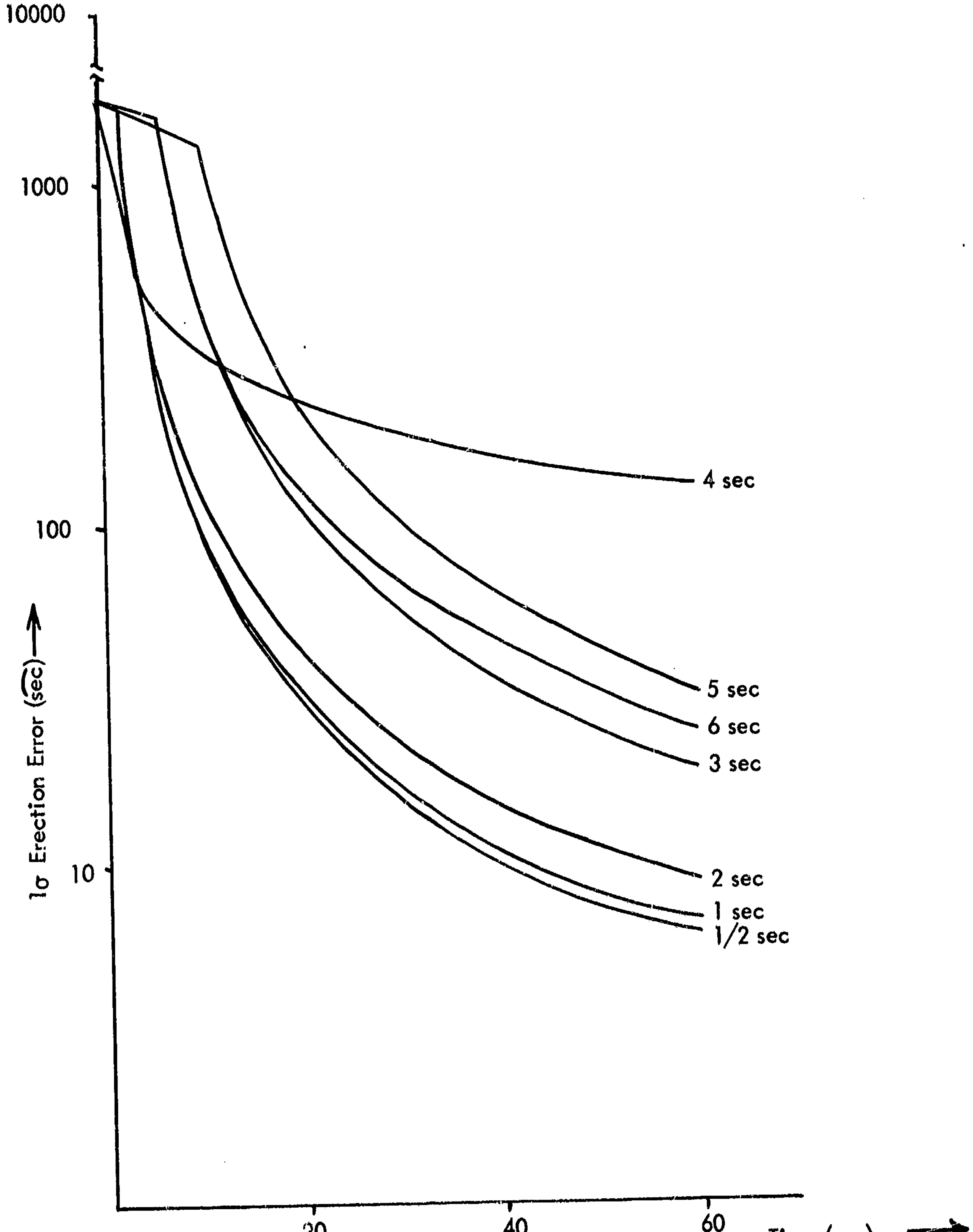
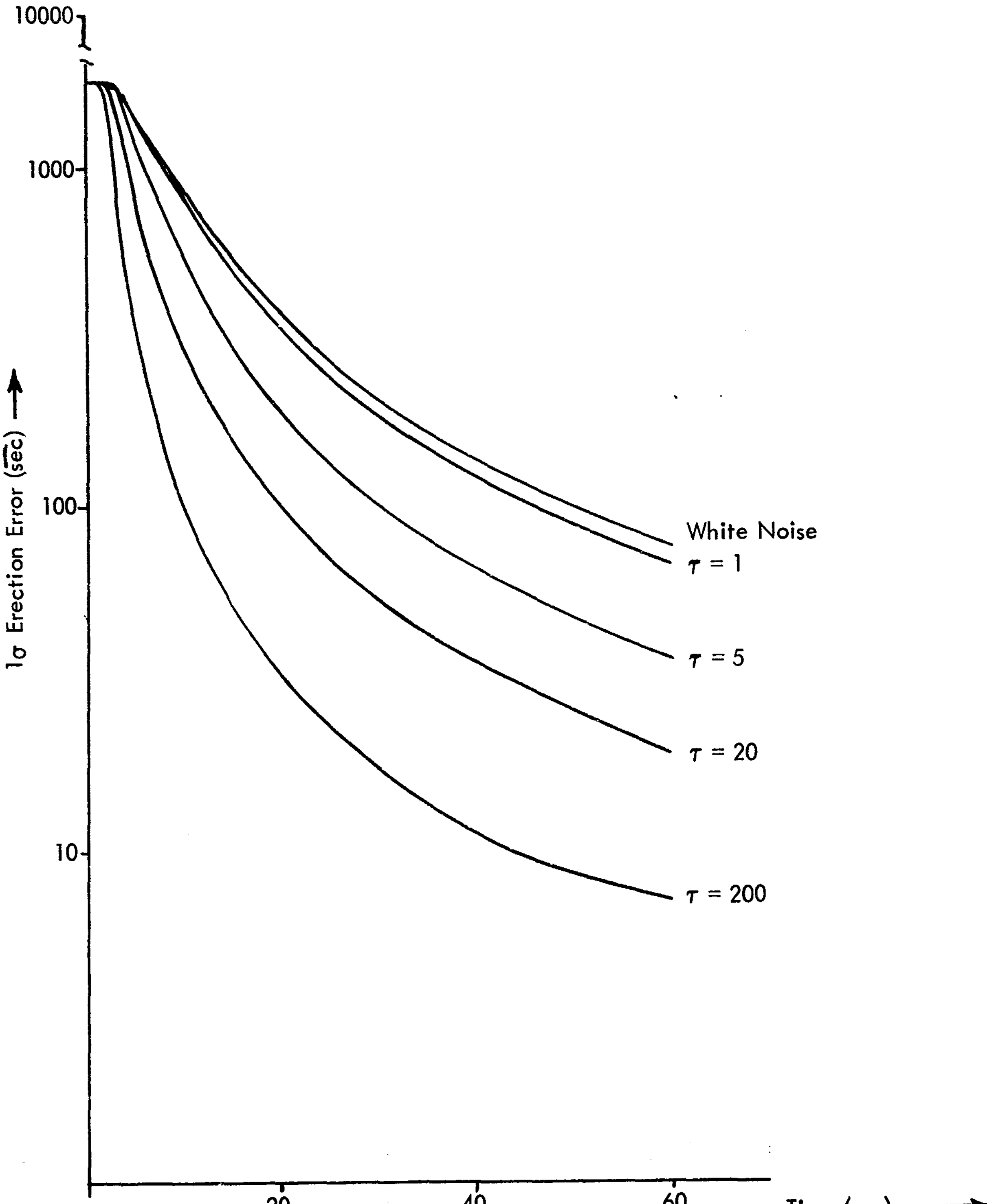


Figure 5. ERECTION ERROR FOR VARIOUS SWAY VELOCITY CORRELATION TIMES



found that the filter was insensitive to errors in the assumed correlation time. When the assumed correlation time was 20 sec. rather than the true correlation time of 200 sec, the degradation in filter performance was negligible. The important parameter is the actual noise correlation time and not the correlation time assumed for the filter calculations.

Since the frequency of the missile sway may not be known exactly, it is of interest to determine the sensitivity of the filter to that parameter. This is shown in Figure 6. In this case, the actual frequency is .25 cps while the number used in the filter calculations is .2 cps. It is seen that the accuracy of the estimate is somewhat poorer, but still acceptable. The performance for an assumed frequency of .3 cps is almost identical to the curve shown in Figure 6 and, therefore, has not been presented.

An alternative method of erecting the analytic coordinate system is to use a least squares filter. The computational requirements of this method are much less than those of the optimal filtering technique. The estimate, of course, takes longer to converge to within acceptable limits. Figure 7 compares the performance of the least squares and the optimal 4-state filters of the case when $\tau = 200$ sec. The optimal filter is significantly better than the least squares filter for this case. Figures 8 and 9 show the same comparison for $\tau = 20$ sec and 5 sec respectively. It is seen that as τ decreases, the least squares filter performance approaches that of the optimal filter. This is to be expected since, for white noise ($\tau = 0$), the least squares filter is optimal.

It is possible to further reduce the number of states in the filter from 4 to 3 by sampling at exactly 1/2 the period of the sway velocity. This happens because some of the terms in the filtering equations are multiplied by sin ωt which is always zero at $\omega = \pi/2$. Figure 10 shows the 3-state filter performance. The sensitivity to using the wrong f in the filter equations is also shown. While the filter performance is acceptable, it is felt that constraining the sampling time to 1/2 the period of the sway velocity is too restrictive. A more flexible filter seems desirable.

The preceding curves were calculated by determining the covariance matrices for the estimation errors using the results of Section III-C-2. Figure 11 shows the results of a sample run from the Erection and Alignment Simulation Program described in Section IV. In Figure 11 the sway velocity noise is assumed to be in a narrow band around a center frequency. Figure 12 shows the results of a run in which the sway velocity noise is assumed to consist of 3 narrow band noises at different center frequencies. The center frequencies for the bands are 1.82 rad/sec, 2.14 rad/sec and 2.32 rad/sec with a correlation time of 200 seconds for each band. The total rms sway velocity is 0.5 m/sec. This run was made to provide assurance that the filter would operate properly with a more realistic noise input.

Figure 6. SENSITIVITY OF 4-STATE FILTER TO ERRORS IN ASSUMED CENTER FREQUENCY OF SWAY VELOCITY

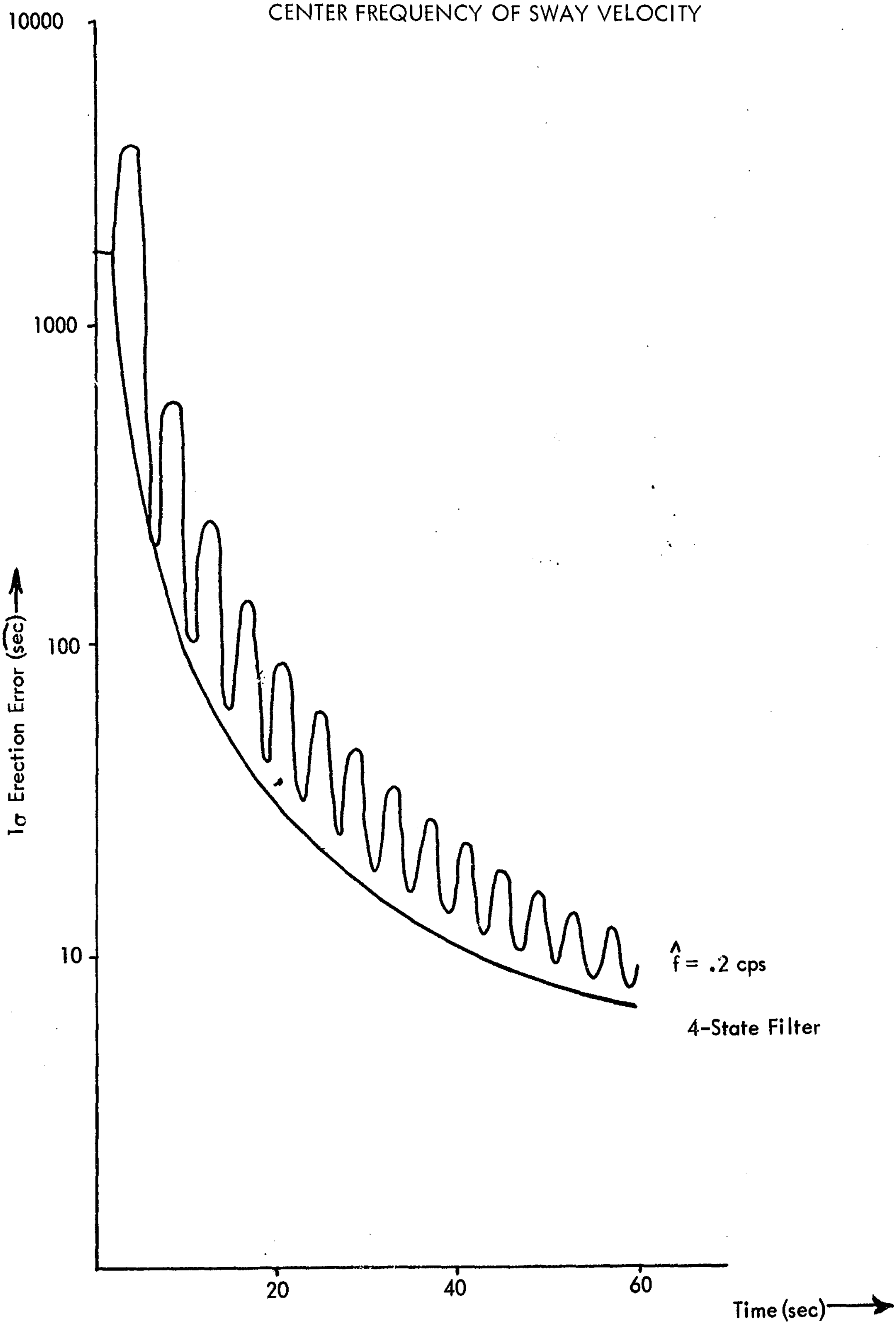


Figure 7. A COMPARISON OF THE 4-STATE AND LEAST SQUARES FILTERS FOR A SWAY VELOCITY CORRELATION TIME OF 200 SEC.

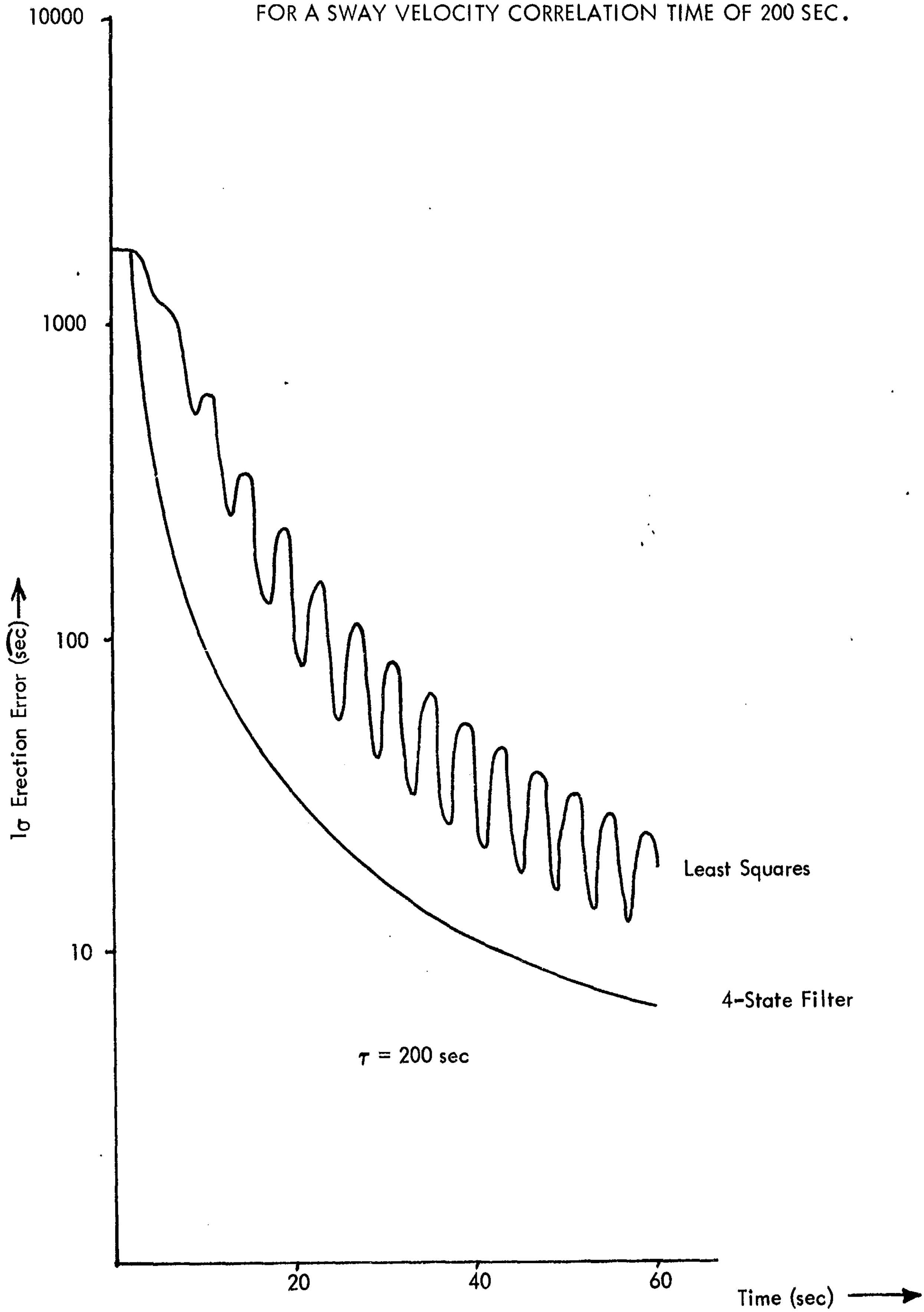


Figure 8. A COMPARISON OF THE 4-STATE AND LEAST SQUARES FILTERS FOR A SWAY VELOCITY CORRELATION TIME OF 20 SEC.

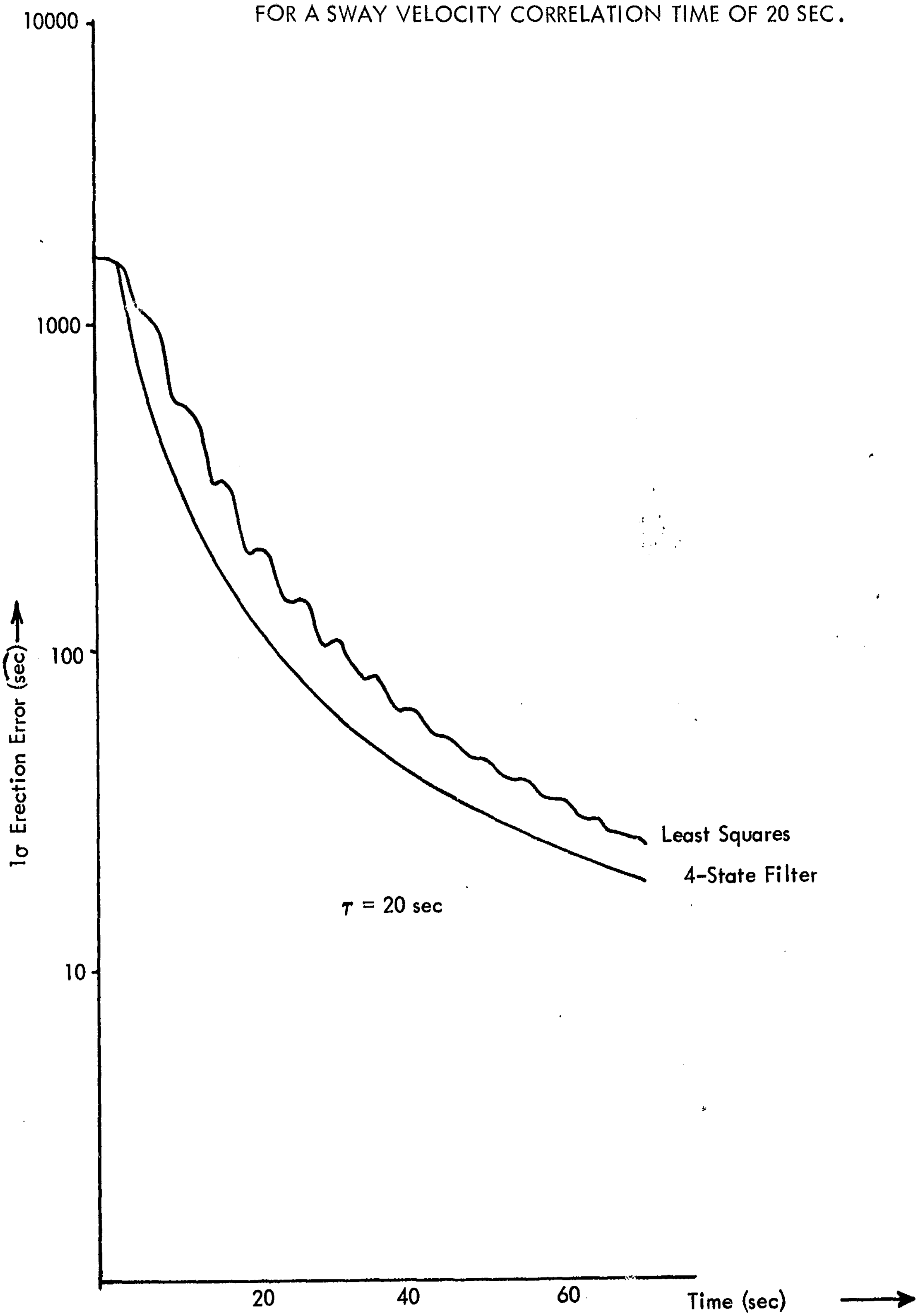


Figure 9. A COMPARISON OF THE 4-STATE AND LEAST SQUARES FILTERS FOR A SWAY VELOCITY CORRELATION TIME OF 5 SEC.

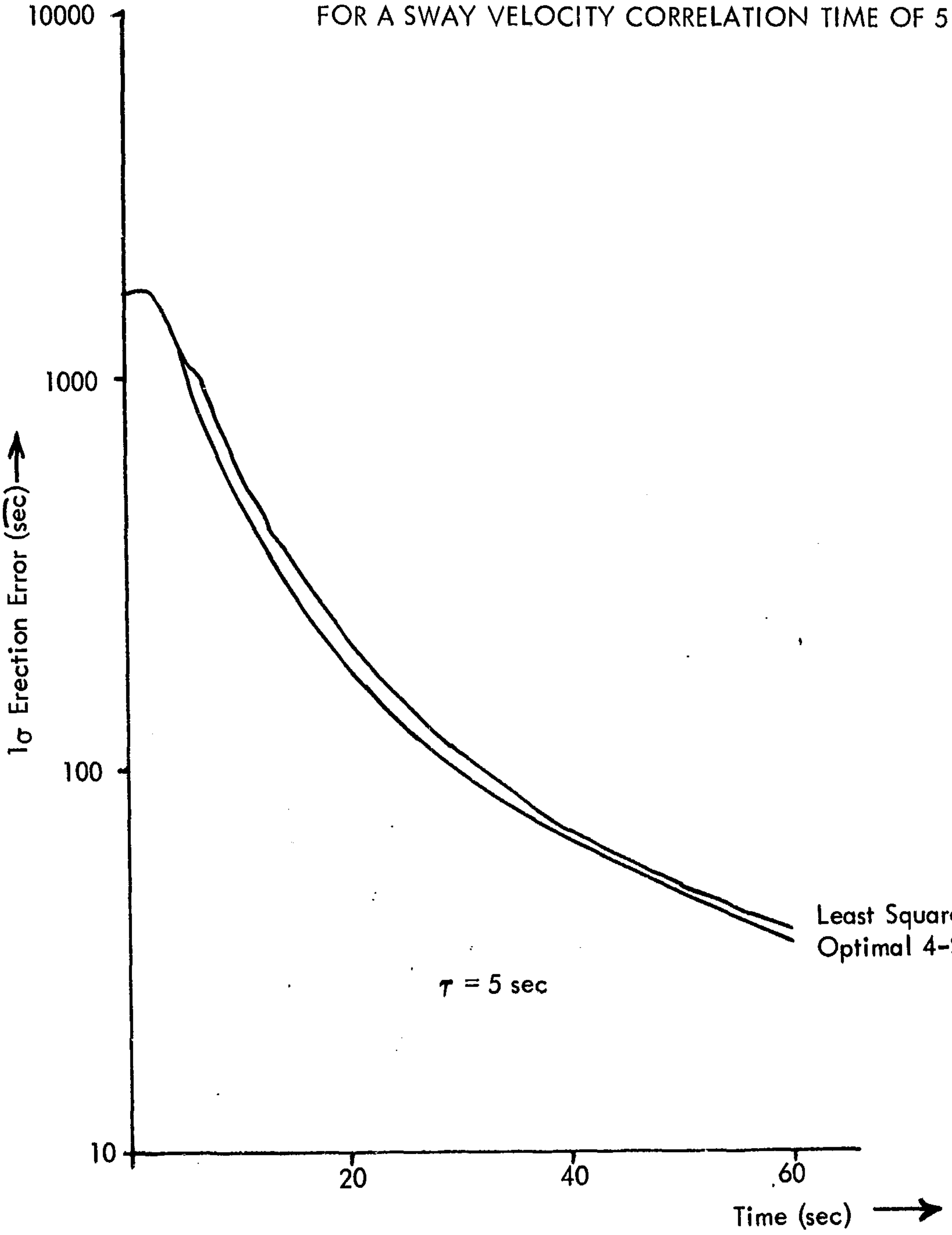
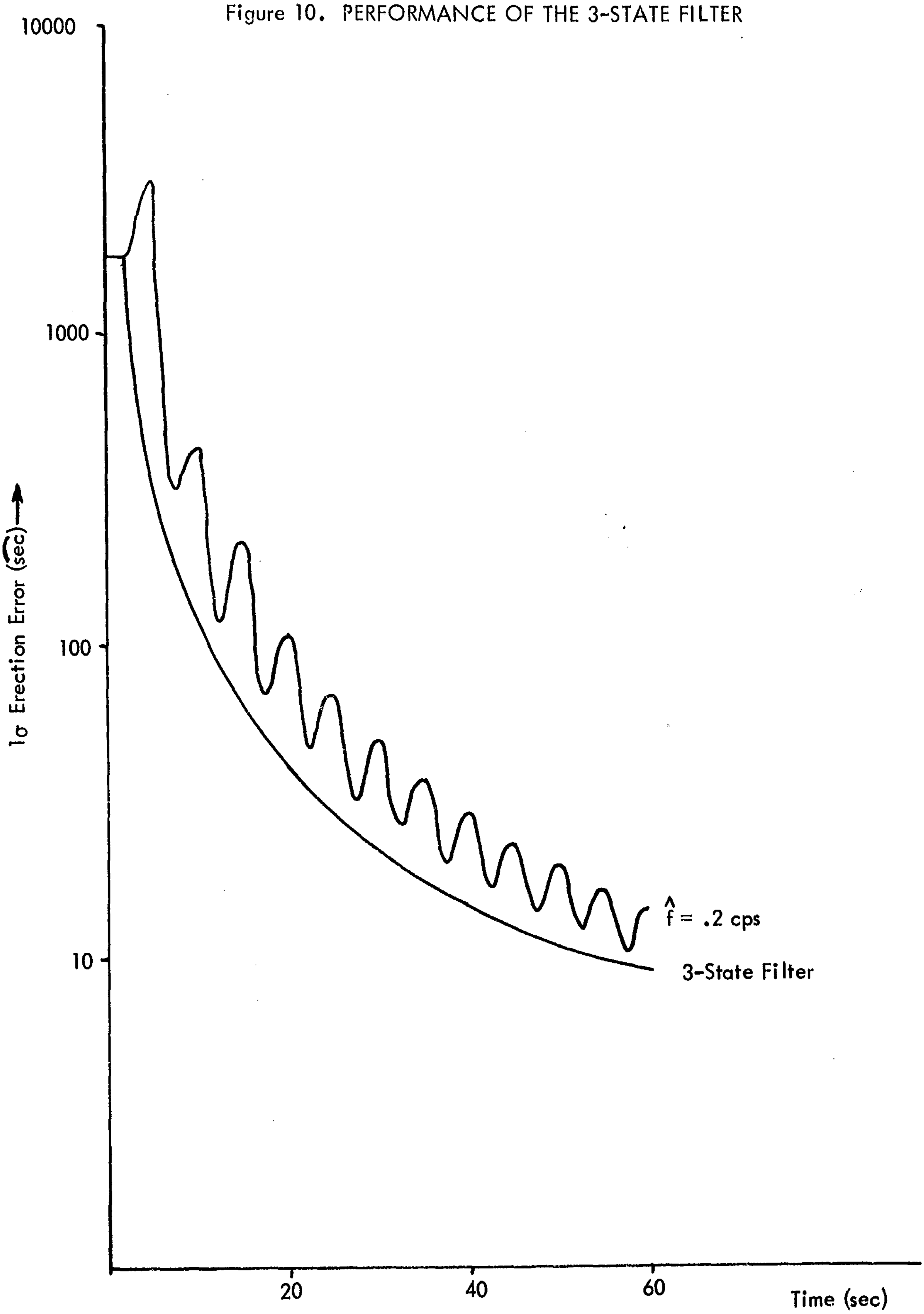


Figure 10. PERFORMANCE OF THE 3-STATE FILTER



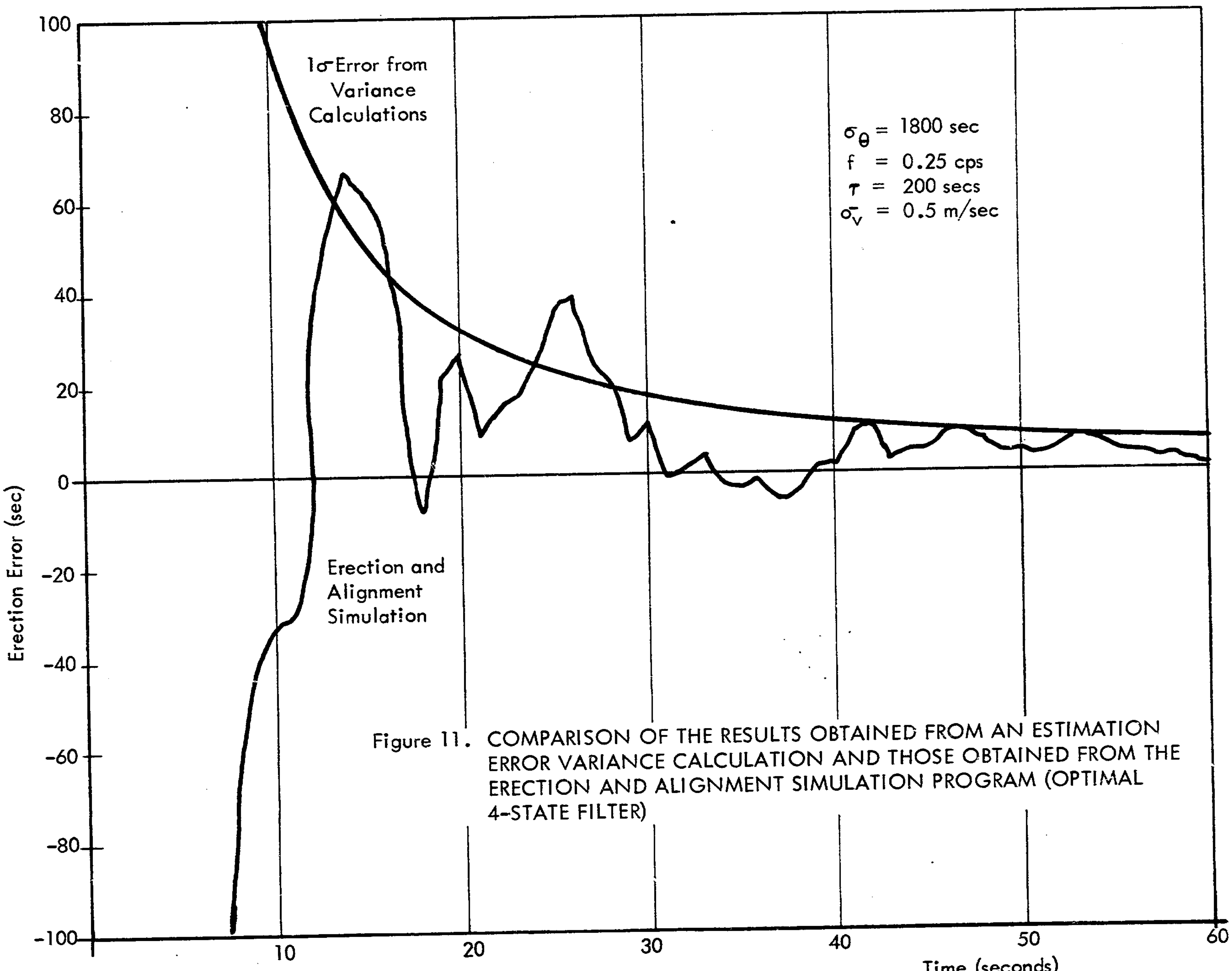


Figure 11. COMPARISON OF THE RESULTS OBTAINED FROM AN ESTIMATION ERROR VARIANCE CALCULATION AND THOSE OBTAINED FROM THE ERECTION AND ALIGNMENT SIMULATION PROGRAM (OPTIMAL 4-STATE FILTER)

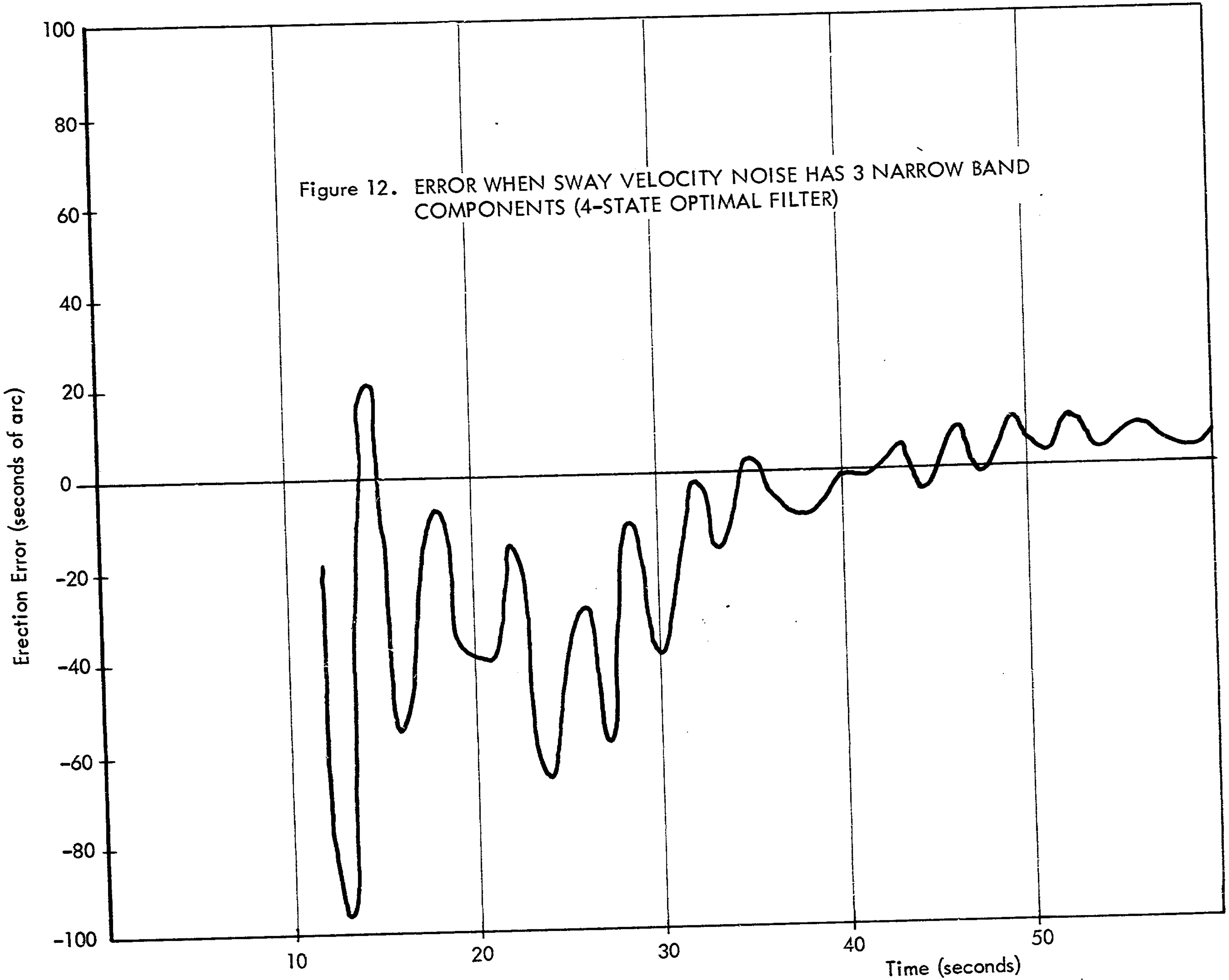


Figure 12. ERROR WHEN SWAY VELOCITY NOISE HAS 3 NARROW BAND COMPONENTS (4-STATE OPTIMAL FILTER)

Figure 13 shows the results of the analysis of a least squares curve fit filter. This analysis was performed by a Monte Carlo computer simulation as described in Section III-D. The curve labelled non integrated data is derived by fitting the velocity data from the accelerometers to a constant plus a linear time term. The curve labelled integrated data is derived by fitting the integral of the velocity data to a quadratic function. The asymptotes on Figure 13 are the errors when the noise is considered to be a sine wave of a single frequency ($\tau = \infty$).

The results of the maximum likelihood filter analysis are contained in Section III-E. These results in general confirm the results of the Kalman filter analysis and are not repeated here.

Computer storage requirements are 636 locations if the optimal 4-state filter is used and 342 locations of the least squares filter is used. These numbers include provision for the azimuth and correction matrix calculations. Execution times are tabulated in Section V. Fixed point calculations with the 23 bit word length of the RCA-110A computer are acceptable.

Certain "rules of thumb" which have been developed to allow extrapolation of the results given in this report to different noise characteristics and filtering times are given below.

a) The rms filtering error is proportional to the rms value of the sway velocity noise.
b) If earth's rate crosscouplings are removed by updating or by using the correction matrix, the filtering error is not a strong function of the initial angle error (See table 2, Section III-D).
c) The filtering error is approximately inversely proportional to the noise center frequency (See Section III-E and Equation 3-135, Section III-D).
d) For short correlation times, the filtering error for both the optimal 4-state and least squares filter is approximately inversely proportional to the three halves power of the filtering time (Error = $\frac{K}{T^{3/2}}$). This is shown by plotting the results of Figure 8 on log-log paper in Figure 14. For longer correlation times, the least squares filter has a filtering error which is more nearly inversely proportional to the square of the filtering time while the 4-state optimal remains inversely proportional to the three halves power of the filtering time. This is shown by plotting the results of Figure 7 on log-log paper in Figure 15. It must be remembered, however, that the curves for the two filters can never intersect so that the least squares filter error must approach being inversely proportional to the three halves power of filtering time as the time increases.

Figure 13. RMS ERROR (98-102 SECONDS AFTER START) USING LEAST SQUARES CURVE FIT FILTER

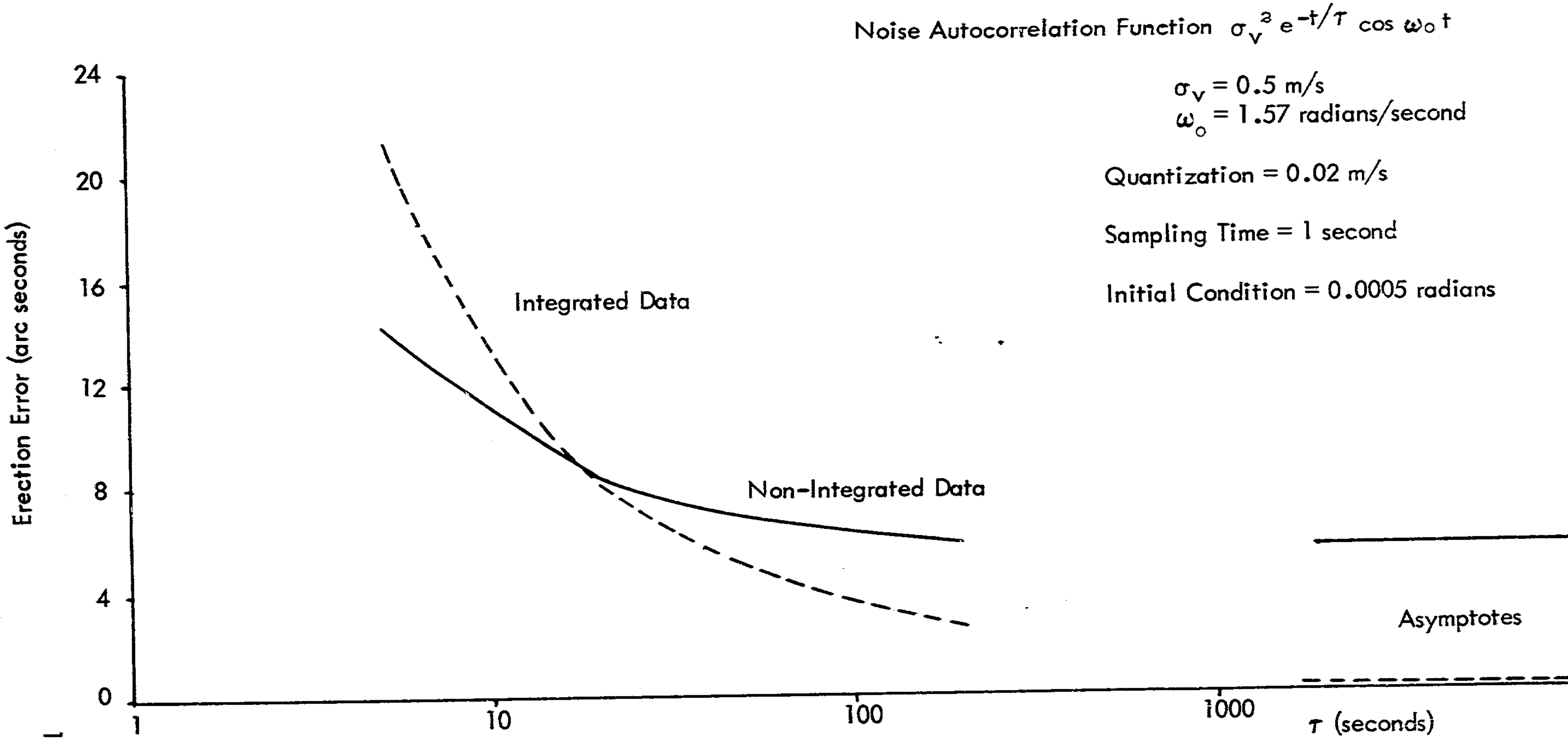


Figure 14. DEPENDENCE OF THE FILTERING ERROR ON FILTERING TIME (τ = 20 SECONDS)

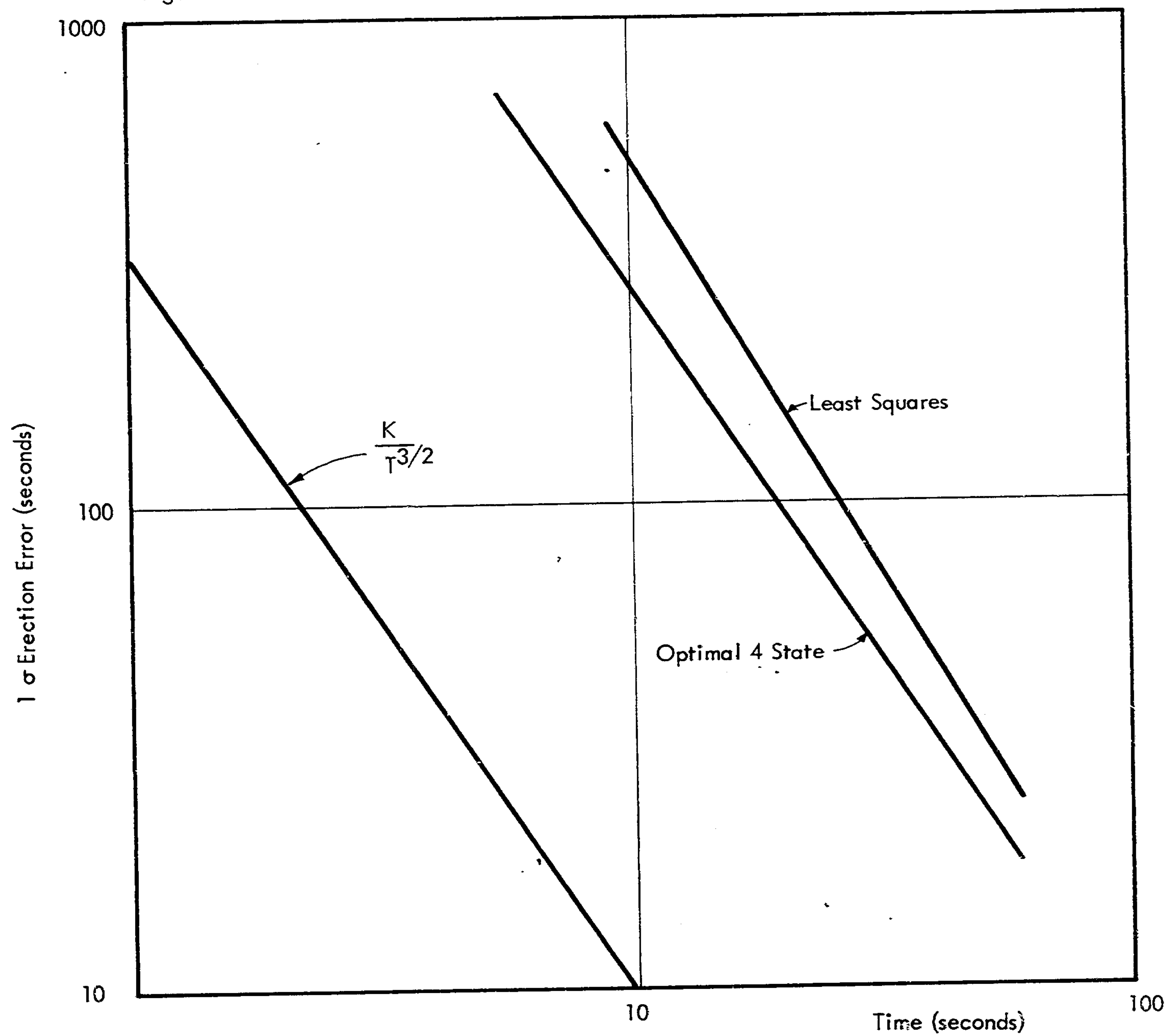
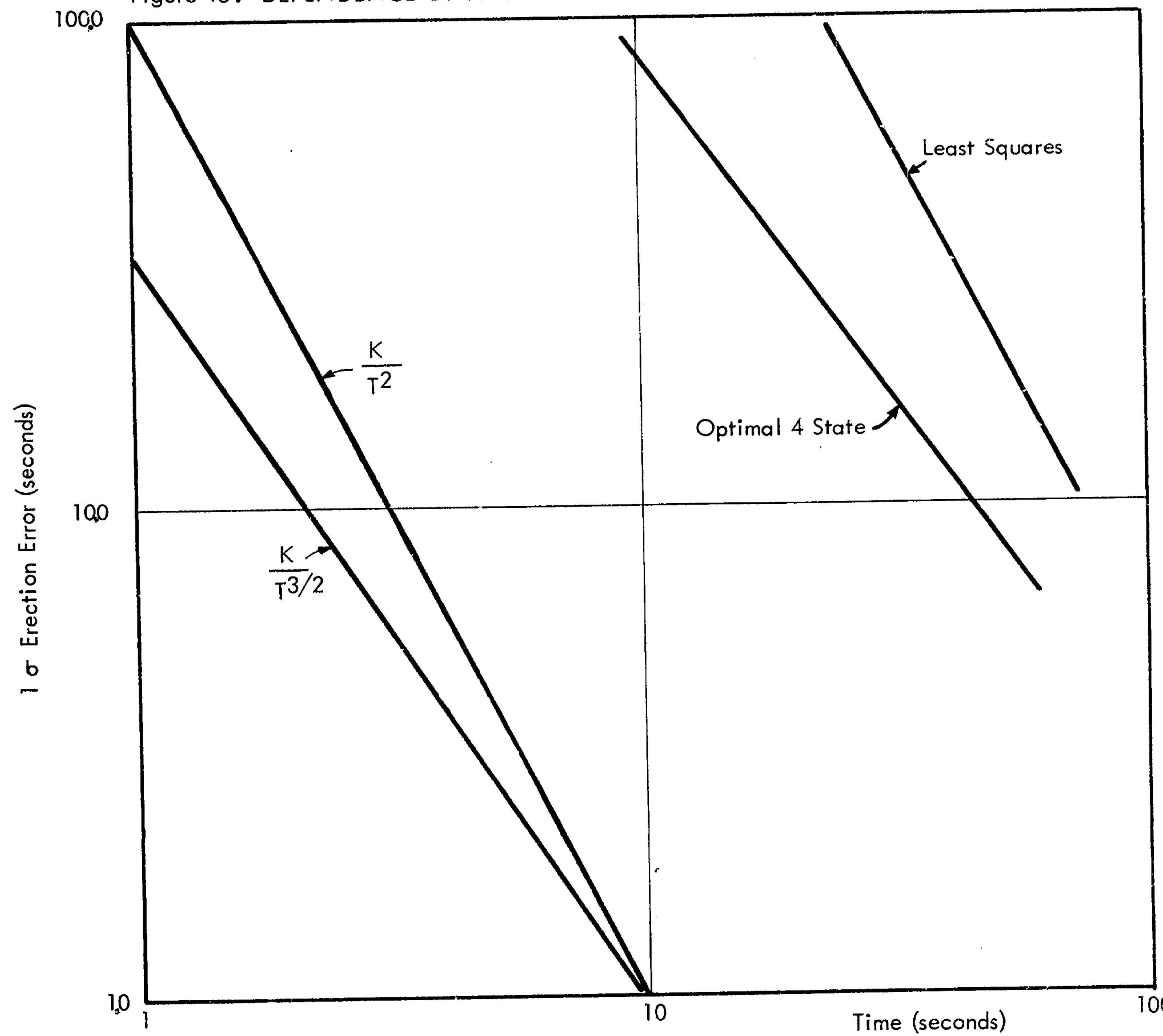
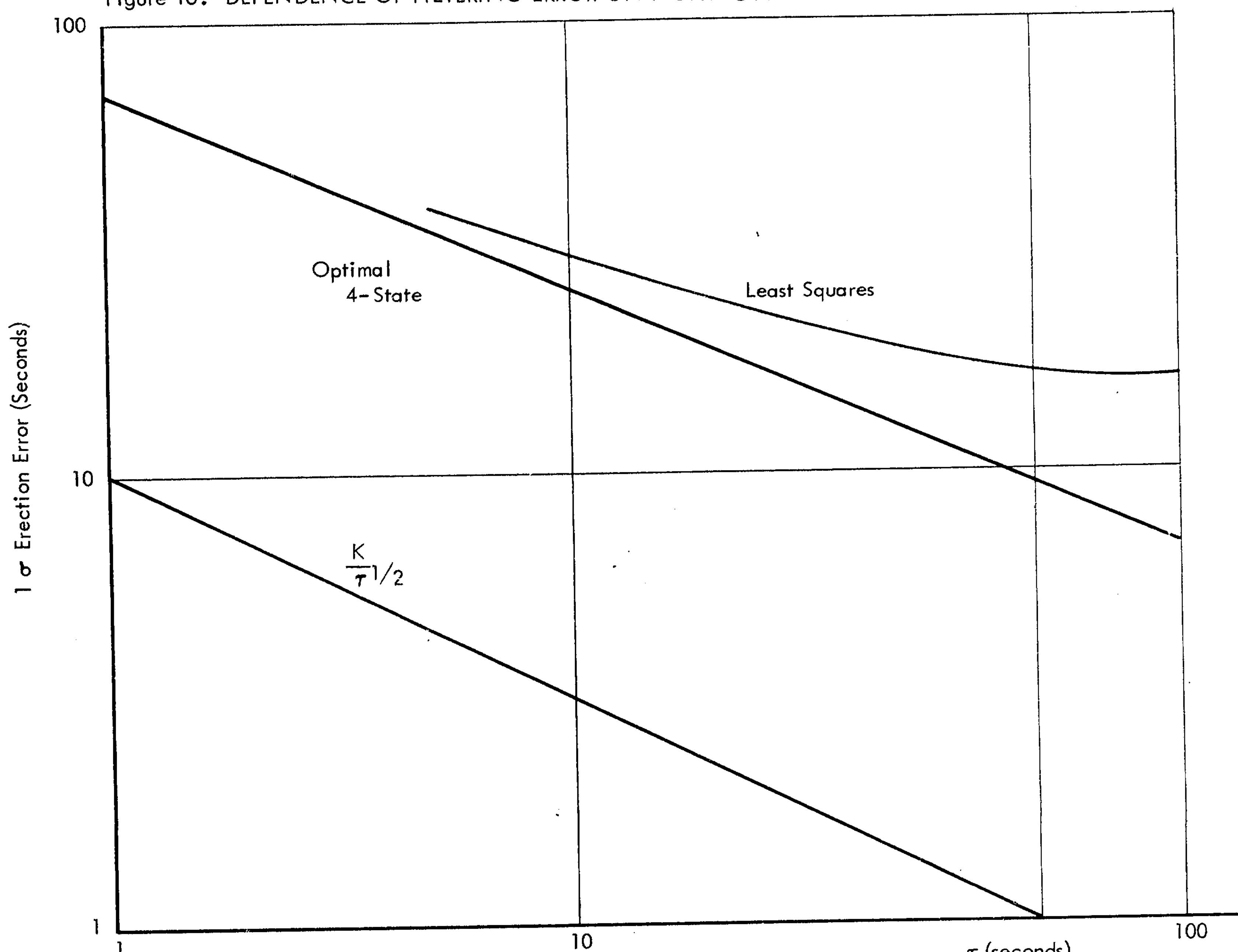


Figure 15. DEPENDENCE OF FILTERING ERROR ON FILTERING TIME (τ = 200 SECONDS)



e) The filtering error for the optimal 4-state filter is approximately inversely proportional to the square root of the sway velocity correlation time (Error $= \dfrac{K}{\tau^{1/2}}$). The least squares filter error only approaches being inversely proportional to the square root of the correlation time at low correlation times, however. For higher correlation time the improvement in performance with increasing correlation time is much less than that of the optimal 4-state filter. This is shown by plotting the results of Figures 7, 8, and 9 at 60 seconds time on log-log paper in Figure 16. This rule obviously cannot be used as τ approaches zero but it does hold at least down to $\tau = 1$ second.

Figure 16. DEPENDENCE OF FILTERING ERROR ON NOISE CORRELATION TIME



III. FILTER ANALYSIS

A. SYSTEM DYNAMICS

Introduction

The equations describing the strapdown erection process are derived in this section. These include the differential equations governing (1) the euler angles describing the misalignment between the earth-fixed launch-site coordinate system (CS) and the strapdown inertial reference CS, (2) the transformed accelerometer outputs, and (3) the missile sway velocity. The differential equations are then converted to a set of first order difference equations since this is the form appropriate for the application of the discrete-time Kalman filter equations. Certain assumptions are made in this analysis. These are:

1. The initial miserection angles are small enough so that the small angle approximations ($\sin \theta = \theta$, $\cos \theta = 1$) are valid.

2. The gyro drift rates are constant.

3. The sway velocity is adequately represented by a narrow-band noise process.

The Differential Equations Governing the Euler Angles Describing the Misalignment Between the Earth-Fixed Launch Site Coordinate System and the Strapdown Inertial Reference Coordinate System

The differential equation describing the direction cosine matrix between two rotating coordinate systems is given by:

$$^{j}\dot{C}_{i} = {}^{j}C_{i}\,\Omega_{j} - \Omega_{j}\,{}^{j}C_{i} \qquad 3\text{-}1$$

Where: $^{j}C_{i}$ is the direction cosine matrix which transforms a vector from CS(i) to CS(j)
Ω_i is the skew symmetric matrix of the angular rates of CS(i).

The angular rate matrix is given by

$$\boldsymbol{\Omega}_i = \begin{bmatrix} 0 & -\omega_{zi} & \omega_{yi} \\ \omega_{zi} & 0 & -\omega_{xi} \\ -\omega_{yi} & \omega_{xi} & 0 \end{bmatrix}$$

Figure 17 shows the convention for angular rates.

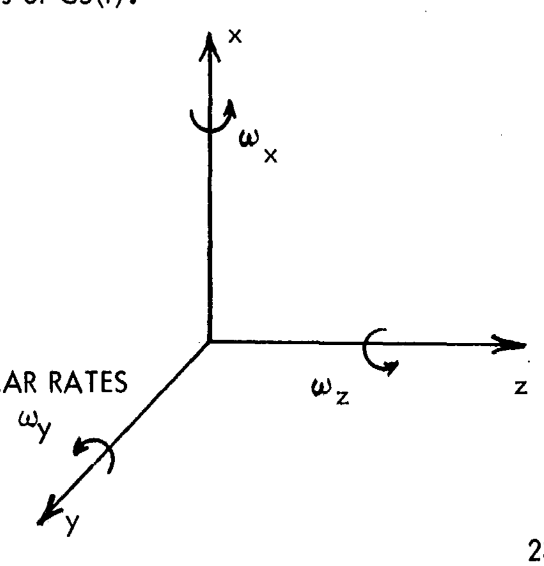


Figure 17.
CONVENTION FOR ANGULAR RATES

The three coordinate systems of interest in this analysis are the earth-fixed launch-site CS, the strapdown inertial reference CS, and the strapdown inertial instrument CS. These are designated as CS(0), CS(1), and CS(2) respectively.

The coördinate system CS(1) is the CS to which the outputs of the body-fixed accelerometers are transformed. This CS is defined by the equation which the CTMC solves, namely:

$$ {}^{1}\dot{C}_2 = {}^{1}C_2\dot{\psi} - \Omega\, {}^{1}C_2 \qquad (3\text{-}2)$$

Where: $\dot{\psi}$ is the matrix of angular rates measured by the gyros
Ω is the matrix of earth's rates in CS (0).

The equation for $^{2}\dot{C}_1 = {}^{2}C_1\,\Omega - \dot{\psi}\,{}^{2}C_1$ (3-3)

The gyro measured rates are in error by the gyro drift. Thus:

$$\dot{\psi} = \dot{\Theta} + D_2 \qquad (3\text{-}4)$$

Where: $\dot{\Theta}$ is the matrix of true angular rates
D_2 is the matrix of gyro drift rates

Substituting equation 3-4 into 3-3 gives:

$$ {}^{2}\dot{C}_1 = {}^{2}C_1\,\Omega = \dot{\Theta}\,{}^{2}C_1 - D_2\,{}^{2}C_1 \qquad (3\text{-}5)$$

The direction cosine matrix $^{0}C_2$ is given by: (3-6)

$$ {}^{0}\dot{C}_2 = {}^{0}C_2\,\dot{\Theta} - \Omega\,{}^{0}C_2 \qquad (3\text{-}6)$$

One can write the direction cosine matrix $^{0}C_1$ as:

$$ {}^{0}C_1 + {}^{0}C_2\,{}^{2}C_1 \qquad (3\text{-}7)$$

Differentiating equation 3-7 yields:

$$ {}^{0}\dot{C}_1 = {}^{0}\dot{C}_2\,{}^{2}C_1 + {}^{0}C_2\,{}^{2}\dot{C}_1 \qquad (3\text{-}8)$$

Upon substituting equations 3-4, 3-5, and 3-7 into 3-8 one obtains:

$$ {}^{0}C_1\,\Omega - \Omega\,{}^{0}C_1 - {}^{0}C_2\,D_2\,{}^{2}C_1 \qquad (3\text{-}9)$$

The last term in equation 3-9 is rewritten as:

$$ {}^{0}C_{2}\ D_{2}\ {}^{2}C_{1} = {}^{0}C_{1}\ {}^{1}C_{2}\ D_{2}\ {}^{1}C_{2} = {}^{0}C_{1}\ D_{1} \tag{3-10} $$

Thus equation 3-9 becomes

$$ {}^{0}\dot{C}_{1} = {}^{0}C_{1}\ \Omega - \Omega\ {}^{0}C_{1} - {}^{0}C_{1}\ D_{1} \tag{3-11} $$

The transformation between CS(0) and CS(1) is defined by the three euler angles α, β, and γ. These are shown below in Figure 18.

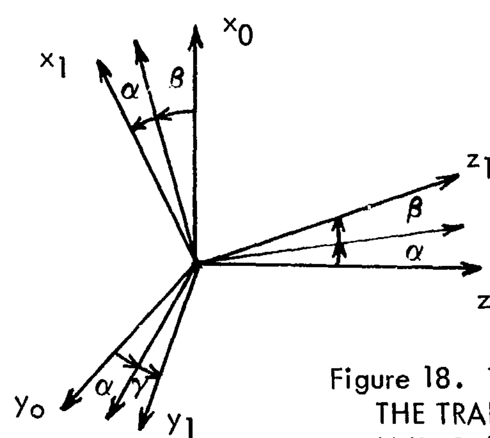


Figure 18. THE EULER ANGLES DEFINING THE TRANSFORMATION BETWEEN CS(0) AND CS(1)

When the small angle approximations are employed the matrix ${}^{0}C_{1}$ can be written as

$$ {}^{0}C_{1} = \begin{bmatrix} 1 & -\gamma & \beta \\ \gamma & 1 & -\alpha \\ -\beta & \alpha & 1 \end{bmatrix} \tag{3-12} $$

Substituting equation 3-12 into equation 3-11 yields:

$$ \left.\begin{aligned} \begin{matrix} 0 & -\dot{\gamma} & \dot{\beta} \\ \dot{\gamma} & 0 & -\dot{\alpha} \\ -\dot{\beta} & \dot{\alpha} & 0 \end{matrix} &= \begin{bmatrix} 1 & -\gamma & \beta \\ \gamma & 1 & -\alpha \\ -\beta & \alpha & 1 \end{bmatrix} \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \\ &- \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \begin{bmatrix} 1 & -\gamma & \beta \\ \gamma & 1 & -\alpha \\ -\beta & \alpha & 1 \end{bmatrix} \\ &- \begin{bmatrix} 1 & -\gamma & \beta \\ \gamma & 1 & -\alpha \\ -\beta & \alpha & 1 \end{bmatrix} \begin{bmatrix} 0 & -d_z & d_y \\ d_z & 0 & -d_x \\ -d_y & d_x & 0 \end{bmatrix} \end{aligned}\right\} \tag{3-13} $$

From equation 3-13 one easily obtains the following three simultaneous differential equations:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} = - \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} - \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} \qquad (3\text{-}14)$$

In matrix-vector notation, this is written as:

$$\dot{\theta} = -\Omega\, \theta \; -d \qquad (3\text{-}15)$$

Where:

$$\theta = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \qquad d = \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}$$

<u>The Differential Equations Governing the Transformed Accelerometer Outputs</u>

The CTMC output gives the integral of acceleration along the coordinates of CS(1). In particular, the outputs along the y and z axes are given by:

$$p_1 \; (t) = \int_{to}^{t} \left[g\, \beta(\tau) + \dot{v}_1 \; (\tau) \right] d\tau \qquad (3\text{-}16)$$

$$p_2 \; (t) = \int_{to}^{t} \left[-g\, \gamma \; (\tau) + \dot{v}_2 \, (\tau) \right] d\tau \qquad (3\text{-}17)$$

Where:

p_1 (t) is the output along z_1 at time t
p_2 (t) is the output along y_1 at time t
$\dot{v}_1$ (t) is the sway acceleration along z_1
$\dot{v}_2$ (t) is the sway acceleration along y_1
g is the value of gravity (m/sec^2)

Differentiating equations 3-16 and 3-17 yields the desired differential equations:

$$\dot{p}_1 = g\, \beta + \dot{v}_1 \qquad (3\text{-}18)$$

$$\dot{p}_2 = -g\, \gamma + \dot{v}_2 \qquad (3\text{-}19)$$

Or

$$\dot{p} = B\; \theta + \dot{v} \qquad (3\text{-}20)$$

Where:

$$p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \qquad \beta = \begin{bmatrix} 0 & g & 0 \\ 0 & 0 & -g \end{bmatrix}$$

The Differential Equations Governing the Missile Sway Velocity

It is assumed that the power spectrum of the sway velocities along the y, and z, axes is adequately described by:

$$S_{V_1}(\omega) = S_{V_2}(\omega) = \sigma_V^2 \left[\frac{1/\tau}{(\omega-\omega_0)^2 + (1/\tau)^2} + \frac{1/\tau}{(\omega+\omega_0)^2 + (1/\tau)^2} \right] \tag{3-21}$$

Where σ_V is the rms sway velocity.

This power spectrum is shown in figure 3-3.

It can be shown also that if n_1 (t) and n_2 (t) are independent random variables with power spectra

$$S_{n_1}(\omega) = S_{n_2}(\omega) = \frac{2\sigma_V^2/\tau}{\omega^2 + (1/\tau)^2} \tag{3-23}$$

Then the random variable

$$v_1(t) = n_1(t) \cos \omega_0 t + n_2(t) \sin \omega_0 t \tag{3-24}$$

will have the power spectrum shown in Figure 19.

Noise with the power spectrum shown in equation 3-23 has the corresponding autocorrelation function:

$$R_{n_1}(t) = R_{n_2}(t) = \sigma_V^2 \; e^{-|t|/\tau} \tag{3-25}$$

Noise with this autocorrelation function can be considered as being the solution to the stochastic differential equation:

$$\dot{n}_1(t) = (-1/\tau) n_1(t) + w_1(t) \tag{3-26}$$

$$\dot{n}_2(t) = (-1/\tau) n_2(t) + w_2(t) \tag{3-27}$$

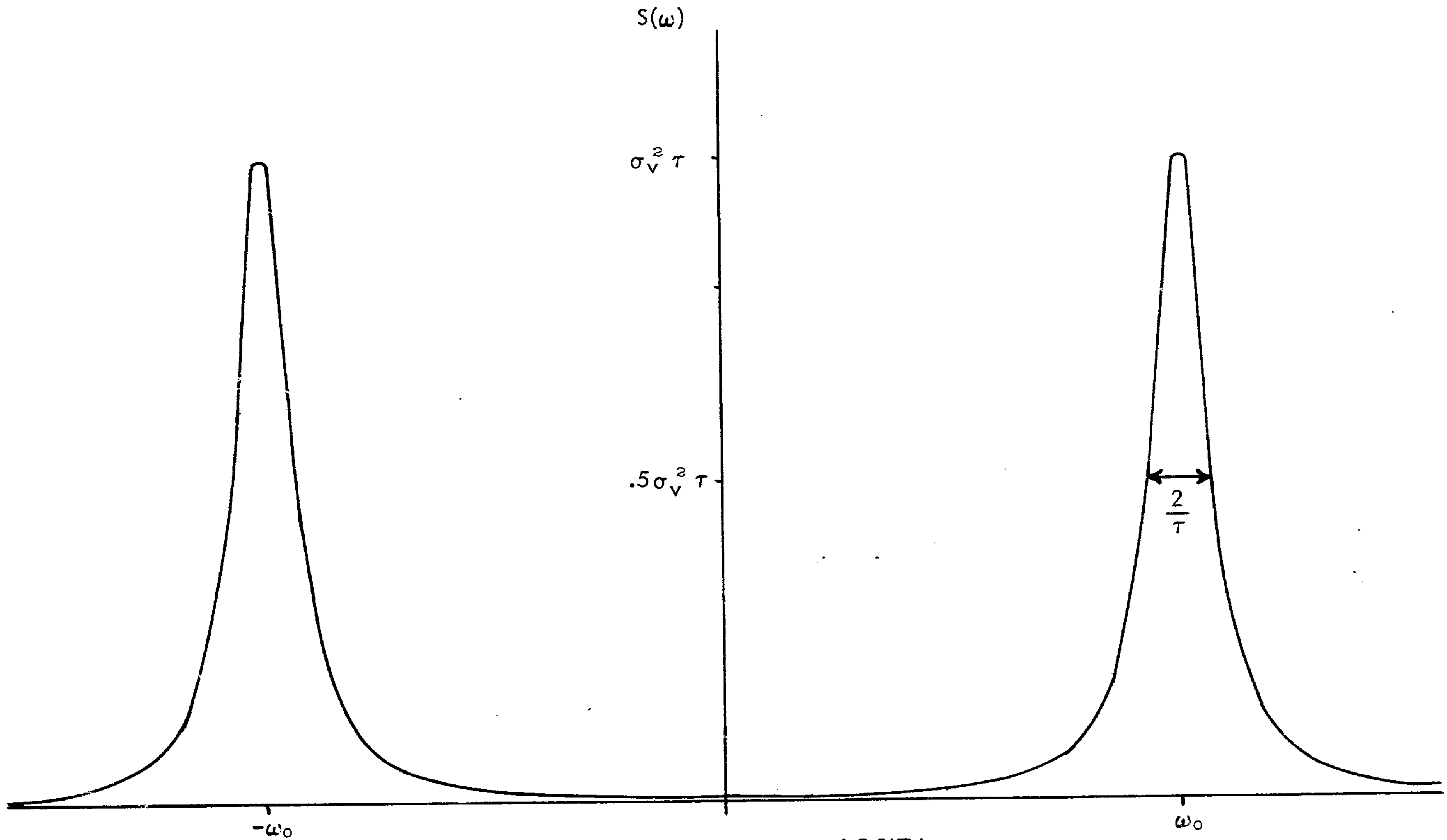


Figure 19. POWER SPECTRUM OF SWAY VELOCITY

Where $w_1(t)$ and $w_2(t)$ are white noise random variables with autocorrelation

$$E\left[w_1(t_1)\, w_1(t_2)\right] = E\left[w_2(t_1)\, w_2(t_2)\right] = \frac{2\sigma_V^2}{\tau}\, \delta(t_1 - t_2) \tag{3-28}$$

$\delta(t_1 - t_2)$ is the dirac delta function.

$v_1(t)$ is the sway velocity along the z axis. For the sway velocity along the y, axis, $v_2(t)$, we have similar equations:

$$v_2(t) = n_3(t) \cos \omega_0 t + n_4(t) \sin \omega_0 t \tag{3-29}$$

$$\dot{n}_3(t) = -1/\tau\, n_3(t) + w_3(t) \tag{3-30}$$

$$\dot{n}_4(t) = -1/\tau\, n_4(t) + w_4(t) \tag{3-31}$$

Where:

$$E\left[w_3(t_1)\, w_3(t_2)\right] = E\left[w_4(t_1)\, w_4(t_2)\right] = \frac{2\sigma_V^2}{\tau}\, \delta(t_1 - t_2) \tag{3-32}$$

In vector notation, then, we have

$$v(t) = C(t)\; n(t) \tag{3-33}$$

$$\dot{n}(t) = An(t) + w(t) \tag{3-34}$$

Where:

$$\dot{v}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \qquad C(t) = \begin{bmatrix} \cos \omega_0 t & \sin \omega_0 t & 0 & 0 \\ 0 & 0 & \cos \omega_0 t & \sin \omega_0 t \end{bmatrix}$$

$$E\left[w(t_1)\, w'(t_2)\right] = \frac{2\sigma_V^2}{\tau}\, I \qquad A = (-1/\tau)\, I$$

Conversion of the Differential Equations to Difference Equations

The differential equation

$$\dot{x}(t) = F\, x(t) + w(t) \tag{3-35}$$

has the solution

$$x(t_i) = \phi(t_i, t_{i-1})\; x(t_{i-1}) + \int_{t_{i-1}}^{t_i} \phi(t_i, \lambda)\; w(\lambda)\; d\lambda \tag{3-36}$$

Where $\emptyset\,(t_i,\ t_{i-1})$ is the matrix exponential

$$\emptyset\ (t_i,\ t_{i-1}) = \exp\left[(t_i - t_{i-1})\ F\right] \tag{3-37}$$

Equation 3-36 can be rewritten

$$x\,(t_i) = \emptyset\,(t_i,\ t_{i-1})\ x\ (t_{i-1}) + u\ (t_i) \tag{3-38}$$

Where:

$$u\,(t_i) = \int_{t_{i-1}}^{t_i} \emptyset\,(t_i,\ \lambda\ )\ w\ (\ \lambda\ )\ d\lambda \tag{3-39}$$

Equation 3-38 is thus the difference equation which is equivalent the differential equation 3-35.

Using this technique, the solution to equation 3-15 is given by:

$$\theta\,(t_i) = \emptyset_{11}\,(t_i,\ t_{i-1})\ \theta\ (t_{i-1}) - \left[\int_{t_{i-1}}^{t_i} \emptyset_{11}\,(t_i,\ \lambda\ )\ d\ \lambda\right]\ d \tag{3-40}$$

Where

$$\emptyset_{11}\,(t_i,\ t_{i-1}) = \exp\left[-\ (t_i - t_{i-1})\ \Omega\right] \tag{3-41}$$

Equation 3-40 can be rewritten

$$\theta(t_i) = \emptyset_{11}\,(t_i,\ t_{i-1})\ \theta\ (t_{i-1}) + \emptyset_{12}\,(t_i,\ t_{i-1})\ d(t_i) \tag{3-42}$$

Where $d(t_i)$ is constant vector and thus satisfies the difference equation

$$d(t_i) = d(t_{i-1}) \tag{3-43}$$

The matrix $\emptyset_{12}\,(t_i,\ t_{i-1})$ is given by:

$$\emptyset_{12}\,(t_i,\ t_{i-1}) = -\int_{t_{i-1}}^{t_i} \emptyset_{11}\,(t_i,\ \lambda\ )\ d\ \lambda \tag{3-44}$$

A solution for $\emptyset_{11}\,(t_i,\ t_{i-1})$ is obtained by using the Cayley-Hamilton method for calculating matrix exponentials. The solution is given by

$$\emptyset_{11}\,(t_i,\ t_{i-1}) = I - \left[\frac{\sin\omega(t_i - t_{i-1})}{\omega}\right]\ \Omega + \left[\frac{1 - \cos\omega(t_i - t_{i-1})}{\omega^2}\right]\ \Omega^2 \tag{3-45}$$

Where ω is the earth's rotation rate.

The matrix $\emptyset_{12}(t_i, t_{i-1})$ is obtained by integrating equation 3-45. The solution is

$$\emptyset_{12}(t_i, t_{i-1}) = -I\ (t_i - t_{i-1}) + \left[\frac{1 - \cos\omega(t_i - t_{i-1})}{\omega^2}\right]\Omega - \left[\frac{(t_i - t_{i-1})}{\omega^2} - \frac{\sin\omega(t_i - t_{i-1})}{\omega^3}\right]\Omega^2 \tag{3-46}$$

Difference equations describing the noise will be derived next. The solution to equation 3-34 is given by:

$$n(t_i) = \emptyset_{44}(t_i, t_{i-1})\, n(t_{i-1}) + \int_{t_{i-1}}^{t_i} \emptyset_{44}(t_i, \lambda)\ w(\lambda)\ d\lambda \tag{3-47}$$

Where:

$$\emptyset_{44}(t_i, t_{i-1}) = \exp\left[(t_i - t_{i-1})\, A\right] = e^{-(t_i - t_{i-1})/\tau}\ I \tag{3-48}$$

Equation 3-47 can be rewritten as

$$n(t_i) = \emptyset_{44}(t_i, t_{i-1})\, n(t_{i-1}) + u(t_i) \tag{3-49}$$

Where:

$$u(t_i) = \int_{t_{i-1}}^{t_i} \emptyset_{44}(t_i, \lambda)\ w(\lambda)\ d\lambda \tag{3-50}$$

$u(t_i)$ is a vector random variable. Its covariance matrix is given by

$$Q(t_i) = E\left[u(t_i)\ u'(t_i)\right]$$

$$= \int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t_i} e^{-(t_i - \lambda_1)/\tau}\ E\left[w(\lambda_1)\ w'(\lambda_2)\right] e^{-(t_i - \lambda_2)/\tau}\ d\lambda_1,\ d\lambda_2 \tag{3-51}$$

Since

$$E\left[w(\lambda_1)\ w'(\lambda_2)\right] = \frac{2\sigma_V^2}{\tau}\ \delta(\lambda_1 - \lambda_2) \tag{3-52}$$

equation 3-51 reduces to a single integral

$$Q(t_i) = \frac{2\sigma_V^2}{\tau} \int_{t_{i-1}}^{t_i} e^{-2(t_i - \lambda_1)/\tau}\ d\lambda_1$$

The solution to this integral is given by

$$Q(t_i) = \sigma_v^2 \left[1 - e^{-2(t_i - t_{i-1})/\tau}\right] \quad (3\text{-}54)$$

Finally, a recursive expression for the accelerometer outputs must be obtained. Equation 3-20 can be integrated to give:

$$p(t_i) = B \int_{t_o}^{t_i} \theta(\lambda)\, d\lambda + v(t_i) - v(t_o) \quad (3\text{-}55)$$

This equation can be rewritten as:

$$p(t_i) = B \int_{t_{i-1}}^{t_i} \theta(\lambda)\, d\lambda + p(t_{i-1}) + v(t_i) - v(t_{i-1}) \quad (3\text{-}56)$$

Substituting equation 3-42 into equation 3-56 gives

$$\left.\begin{aligned} p(t_i) = {} & B \int_{t_{i-1}}^{t_i} \phi_{11}(t_i, \lambda)\, d\lambda \; \theta(t_{i-1}) \\ & + B \int_{t_{i-1}}^{t_i} \phi_{12}(t_i, \lambda)\, d\lambda \; d(t_{i-1}) \\ & + p(t_{i-1}) + v(t_i) - v(t_{i-1}) \end{aligned}\right\} \quad (3\text{-}57)$$

This equation can be rewritten as

$$\begin{aligned} p(t_i) = {} & \phi_{31}(t_i, t_{i-1})\, \theta(t_{i-1}) + \phi_{32}(t_i, t_{i-1})\, d(t_{i-1}) \\ & + p(t_{i-1}) + v(t_i) - v(t_{i-1}) \end{aligned} \quad (3\text{-}58)$$

Where:

$$\begin{aligned} \phi_{31}(t_i, t_{i-1}) &= B \int_{t_{i-1}}^{t_i} \phi_{11}(t_i, \lambda)\, d\lambda \\ &= -B\, \phi_{12}(t_i, t_{i-1}) \end{aligned} \quad (3\text{-}59)$$

$$\phi_{32}(t_i, t_{i-1}) = B \int_{t_{i-1}}^{t_i} \phi_{12}(t_i, \lambda)\, d\lambda$$

$$= B\left\{\left[\frac{-(t_i - t_{i-1})^2}{2}\right] I + \left[\frac{(t_i - t_{i-1})}{\omega^2} - \frac{\sin \omega(t_i - t_{i-1})}{\omega^3}\right]\Omega - \left[\frac{(t_i - t_{i-1})^2}{2\,\omega^2} - \frac{1 - \cos \omega(t_i - t_{i-1})}{\omega^4}\right]\Omega^2\right\} \qquad (3\text{-}60)$$

Equations 3-33 and 3-49 are substituted into equation 3-58 to give

$$p(t_i) = \phi_{31}(t_i, t_{i-1})\,\theta(t_{i-1}) + \phi_{32}(t_i, t_{i-1})\, d(t_{i-1}) + p(t_{i-1}) + \phi_{34}(t_i, t_{i-1})\, n(t_{i-1}) + C(t_i)\, u(t_i) \qquad (3\text{-}61)$$

Where:

$$\phi_{34}(t_i, t_{i-1}) = C(t_i)\,\phi_{44}(t_i, t_{i-1}) - C(t_{i-1}) \qquad (3\text{-}62)$$

Combining equations 3-42, 3-43, 3-49 and 3-61 gives

$$\begin{bmatrix}\theta(t_i)\\ d(t_i)\\ p(t_i)\\ n(t_i)\end{bmatrix} = \begin{bmatrix}\phi_{11}(t_i, t_{i-1}) & \phi_{12}(t_i, t_{i-1}) & 0 & 0\\ 0 & I & 0 & 0\\ \phi_{31}(t_i, t_{i-1}) & \phi_{32}(t_i, t_{i-1}) & I & \phi_{34}(t_i, t_{i-1})\\ 0 & 0 & 0 & \phi_{44}(t_i, t_{i-1})\end{bmatrix}\begin{bmatrix}\theta(t_{i-1})\\ d(t_{i-1})\\ p(t_{i-1})\\ n(t_{i-1})\end{bmatrix} + \begin{bmatrix}0\\ 0\\ C(t_i)\\ I\end{bmatrix}\begin{bmatrix}u(t_i)\end{bmatrix} \qquad (3\text{-}63)$$

This equation is written:

$$x(t_i) = \phi(t_i, t_{i-1})\, x(t_{i-1}) + G(t_i)\, u(t_i) \qquad (3\text{-}64)$$

Where:

$$x(t_i) = \begin{bmatrix}\theta(t_i)\\ d(t_i)\\ p(t_i)\\ n(t_i)\end{bmatrix} \qquad \phi(t_i, t_{i-1}) = \begin{bmatrix}\phi_{11}(t_i, t_{i-1}) & \phi_{12}(t_i, t_{i-1}) & 0 & 0\\ 0 & I & 0 & 0\\ \phi_{31}(t_i, t_{i-1}) & \phi_{32}(t_i, t_{i-1}) & I & \phi_{34}(t_i, t_{i-1})\\ 0 & 0 & 0 & \phi_{44}(t_i, t_{i-1})\end{bmatrix} \qquad (3\text{-}65)$$

The measurement consists of the two transormed accelerometer outputs. There is noise on the measurement due to quantization of the accelerometer outputs. The measurement equation is given by:

$$y(t_i) = H(t_i) \times (t_i) + r(t_i) \tag{3-66}$$

Where:

$$H(t_i) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3-67}$$

$r(t_i)$ is the noise due to quantization. Its covariance matrix is given by

$$R(t_i) = E\left[r(t_i)\ r'(t_i)\right] = \frac{\Delta^2}{12} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Where Δ is the quantization level.

When a theodolite measurement is taken and processed, the resulting quantity is the azimuth misalignment plus an error which is equal to the levelling error about the projection of the LOS in the horizontal plane multiplied by the tangent of the elevation angle. Thus when a theodolite measurement is taken, the $H(t_i)$ matrix is given

$$H(t_i) = \begin{bmatrix} 1 & \tan\gamma_o \sin\zeta & \tan\gamma_o \cos\zeta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3-69}$$

Where ζ is the angle between the projection of the LOS in the horizontal plane and the z_i axis.

In this section, a description of the system dynamics has been derived. The description is in the form of simultaneous first order difference equations. These are presented in matrix form by equation 3-64. The measurements on the system are described by equation 3-66. In addition expressions for the various covariance matrices needed for the filter equations have been obtained.

B. OPTIMAL FILTER EQUATIONS

In part A of section III, the system equations were derived. They are repeated here for convenience.

$$x(t_i) = \emptyset(t_i, t_{i-1}) \times (t_{i-1}) + G(t_i)\ u(t_i) \tag{3-70}$$

$$y(t_i) = H\ (t_i) \times (t_i) + r\ (t_i) \tag{3-71}$$

The well known Kalman Filter equations can be applied to systems with this description. These equations give an estimate of the state vector $x(t_i)$, based on the measurements $\{ y(t_1) -- y(t_i) \}$, which is optimal in the mean square sense. The filter is described by the following difference equations:

$$\hat{x}(t_i, t_{i-1}) = \phi(t_i, t_{i-1}) \hat{x}(t_{i-1}) \tag{3-72}$$

$$\hat{x}(t_i) = \hat{x}(t_i, t_{i-1}) + K(t_i) \left[y(t_i) - H(t_i) \hat{x}(t_i, t_{i-1}) \right] \tag{3-73}$$

The optimal filter gains $K(t_i)$ are calculated from the following set of difference equations:

$$P(t_i, t_{i-1}) = \phi(t_i, t_{i-1}) P(t_{i-1}) \phi(t_i, t_{i-1}) + G(t_i) Q(t_i) G'(t_i) \tag{3-74}$$

$$K(t_i) = P(t_i, t_{i-1}) H'(t_i) \left[H(t_i) P(t_i) H'(t_i) + R(t_i) \right]^{-1} \tag{3-75}$$

$$P(t_i) = \left[I - K(t_i) H(t_i) \right] P(t_i, t_{i-1}) \tag{3-76}$$

Where:

$\hat{x}(t_i)$ is the optimal estimate of $x(t_i)$ given the measurements $\{ y(t_1), y(t_2), -- y(t_i) \}$

$\hat{x}(t_i, t_{i-1})$ is the optimal estimate of $x(t_i)$ given the measurements $\{ y(t_1), y(t_2), -- y(t_{i-1}) \}$

$P(t_i)$ is the covariance matrix of the error in the estimate $\hat{x}(t_i)$

$P(t_i, t_{i-1})$ is the covariance matrix of the error in the estimate $\hat{x}(t_i, t_{i-1})$

The above equations describe the optimal filter. A direct application of these equations could be done. It is desirable, however, to simplify and decouple the equations to reduce the computational requirements. The performance of the optimal filter is described in section II. It is also demonstrated that the degredation caused by certain simplifications is negligible.

These simplifications will be investigated in the following section.

C. SUBOPTIMAL FILTER EQUATIONS

Introduction

The filter equations presented in the previous section are rather complex. They involve the solution of 12 simultaneous difference equations and hence multiplication of

12 x 12 matrices. It would be desirable, then, to investigate the possibility of simplifying these equations. In order to determine if the simplifications are acceptable, one must calculate the degradation in filter performance caused by them. In addition, the sensitivity of the filter to errors in the noise model should be investigated. The equations required to evaluate the suboptimal filter performance are derived in this section. The approximations to the system dynamics which are used in the suboptimal filter are also presented. A computer program has been written which solves these equations. Results from this program are given in section II.

Evaluation of Suboptimal Filter Performance

The equations describing the system dynamics (equations 3-64 and 3-65) are repeated here for convenience.

$$x(t_i) = \phi(t_i, t_{i-1})\, x(t_{i-1}) + G(t_i)\, u(t_i) \qquad (3\text{-}77)$$

$$y(t_i) = H(t_i)\, x(t_i) + r(t_i)$$

A fairly general model for the suboptimal filter equations is given by:

$$\hat{x}_s(t_i, t_{i-1}) = \phi_s(t_i, t_{i-1})\, \hat{x}_s(t_{i-1}) \qquad (3\text{-}79)$$

$$\hat{x}_s(t_i) = \hat{x}_s(t_i, t_{i-1}) + K_s(t_i)\left[y(t_i) - H(t_i)\, \hat{x}_s(t_i, t_{i-1})\right] \qquad (3\text{-}80)$$

Where $K_s(t_i)$ is obtained from:

$$P_s(t_i, t_{i-1}) = \phi_s(t_i, t_{i-1})\, P_s(t_{i-1})\, \phi'_s(t_i, t_{i-1}) + G_s(t_i)\, Q_s(t_i)\, G'_s(t_i) \qquad (3\text{-}81)$$

$$K_s(t_i) = P_s(t_i, t_{i-1})\, H'(t_i)\left[H(t_i)\, P(t_i)\, H'(t_i) + R(t_i)\right]^{-1} \qquad (3\text{-}82)$$

$$P_s(t_i) = \left[I - K(t_i)\, H(t_i)\right] P(t_i, t_{i-1}) \qquad (3\text{-}83)$$

The following covariance matrices are defined:

$$\begin{aligned}
P_1(t_i) &= E\left[x(t_i)\, x'(t_i)\right] \\
P_2(t_i, t_{i-1}) &= E\left[\hat{x}_s(t_i, t_{i-1})\right] \\
P_2(t_i) &= E\left[\hat{x}_s(t_i)\, x'(t_i)\right] \\
P_3(t_i, t_{i-1}) &= E\left[\hat{x}_s(t_i, t_{i-1})\, \hat{x}_s'(t_i, t_{i-1})\right] \\
P_3(t_i) &= E\left[\hat{x}_s(t_i)\, \hat{x}_s'(t_i)\right]
\end{aligned}$$

The covariance matrix of estimate errors is given by

$$\Gamma(t_i) = E\left\{\left[x(t_i) - \hat{x}_s(t_i)\right]\left[x(t_i) - \hat{x}_s(t_i)\right]'\right\} \qquad (3\text{-}84)$$

Equation 3-84 can be written in terms of the covariance matrices defined above.

$$\Gamma(t_i) = P_1(t_i) - P_2(t_i) - P'_2(t_i) + P_3(t_i) \tag{3-85}$$

By multiplying equation 3-77 by its transpose and taking expected values of both sides one immediately obtains:

$$P_1(t_i) = \emptyset(t_i, t_{i-1})\, P_1(t_{i-1})\, \emptyset'(t_i, t_{i-1}) + G(t_i)\, Q(t_i)\, G'(t_i) \tag{3-86}$$

Multiplying equation 3-79 by the transpose of equation 3-77 and taking expected values yields:

$$P_2(t_i, t_{i-1}) = \emptyset_s(t_i, t_{i-1})\, P_2(t_{i-1})\, \emptyset'(t_i, t_{i-1}) \tag{3-87}$$

The equation for $P_2(t_i)$ is obtained by multiplying equation 3-80 by $x'(t_i)$. It is given by:

$$P_2(t_i) = \left[I - K_s H(t_i)\right] P_2(t_i, t_{i-1}) + K_s H(t_i)\, P_1(t_i) \tag{3-88}$$

Multiplying equation 3-79 by its transpose and taking expected values gives:

$$P_3(t_i, t_{i-1}) = \emptyset_s(t_i, t_{i-1})\, P_3(t_{i-1})\, \emptyset_s'(t_i, t_{i-1}) \tag{3-89}$$

The equation for $P_3(t_i)$ is obtained by multiplying equation 3-80 by its transpose and taking expected values. After some algebraic manipulation one obtains:

$$\begin{aligned} P_3(t_i) = {} & P_3(t_i, t_{i-1}) + K_s(t_i)\, H(t_i) \left[P_2'(t_i) - P_3(t_i, t_{i-1})\right] \\ & + \left[P_2(t_i) - P_3(t_i, t_{i-1})\right] H'(t_i)\, K_s'(t_i) \\ & + K_s(t_i)\, H(t_i) \left[P_3(t_i, t_{i-1}) - P_1(t_i)\right] H'(t_i)\, K_s'(t_i) \\ & + K_s(t_i)\, R(t_i)\, K_s'(t_i) \end{aligned} \tag{3-90}$$

This completes the derivation of the equations necessary to compute the covariance matrix of estimation errors. The computer program solves equations 3-85 through 3-90 in addition to computing the suboptimal filter gains.

Approximations to System Dynamics

If earth's rotation rate is neglected ($\omega = 0$), the system dynamics are considerably simplified.

The state transition matrix becomes

$$\emptyset(t_i, t_{i-1}) = \begin{bmatrix} I & -(t_i - t_{i-1})I & 0 & 0 \\ 0 & I & 0 & 0 \\ B(t_i - t_{i-1} & -B\,(t_i - t_{i-1}) & I & \emptyset_{34}(t_i, t_{i-1}) \\ 0 & 0 & 0 & \emptyset_{44}(t_i, t_{i-1}) \end{bmatrix} \tag{3-91}$$

The result of this simplification is that the equations for α, β, and γ become coupled. If the drifts are neglected, a further simplification results. The equations describing β are given by

$$\begin{bmatrix} \beta(t_i) \\ P_1(t_i) \\ n_1(t_i) \\ n_2(t_i) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ g(t_i - t_{i-1}) & 1 & \emptyset_{23} & \emptyset_{24} \\ 0 & 0 & \emptyset_{33} & 0 \\ 0 & 0 & 0 & \emptyset_{44} \end{bmatrix} \begin{bmatrix} \beta(t_{i-1}) \\ p(t_{i-1}) \\ n_1(t_{i-1}) \\ n_2(t_{i-1}) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ G_{21} & G_{22} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} u_1(t_i) \\ u_2(t_i) \end{bmatrix} \tag{3-92}$$

Where:

$$\begin{aligned}
\emptyset_{33} &= \emptyset_{44} = \exp - (t_i - t_{i-1})/\tau \\
\emptyset_{23} &= \emptyset_{33} \;\; \cos \omega_0 t_i - \cos \omega_0 t_{i-1} \\
\emptyset_{24} &= \emptyset_{33} \;\; \sin \omega_0 t_i - \sin \omega_0 t_{i-1} \\
G_{21} &= \cos \omega_0 t_i \\
G_{22} &= \sin \omega_0 t_i
\end{aligned}$$

The equations describing $\gamma(t_i)$ are of almost the identical form with a sign difference. This sign difference can be eliminated by defining $p_2(t_i)$ as minus the accelerometer output along the y_2 axis.

The equation for α is simply

$$\alpha(t_i) = \alpha(t_{i-1}) \tag{3-93}$$

The filter corresponding to these uncoupled dynamics is also uncoupled.

This greatly reduces the computational requirements. The filter equations corresponding to these simplifications are given by:

$$\hat{x}_1(t_i, t_{i-1}) = \emptyset^*(t_i, t_{i-1})\,\hat{x}_1(t_{i-1}) \tag{3-94}$$

$$\hat{x}_2(t_i, t_{i-1}) = \emptyset^*(t_i, t_{i-1})\,\hat{x}_2(t_{i-1}) \tag{3-95}$$

$$\hat{x}_1(t_i) = \hat{x}_1(t_i, t_{i-1}) + K^*(t_i)\left[p_1(t_i) - H^*(t_i)\,\hat{x}_1(t_i, t_{i-1})\right] \tag{3-96}$$

$$\hat{x}_2(t_i) = \hat{x}_2(t_i, t_{i-1}) + K^*(t_i)\left[-p_2(t_i) - H^*(t_i)\,\hat{x}_2(t_i, t_{i-1})\right] \tag{3-97}$$

$$P^*(t_i, t_{i-1}) = \emptyset^*(t_i, t_{i-1})\,P^*(t_{i-1})\,\emptyset^{*\prime}(t_i, t_{i-1}) + G^*(t_i)\,Q^*(t_i)\,G^{*\prime}(t_i) \tag{3-98}$$

$$K^*(t_i) = P^*(t_i, t_{i-1})\,H^{*\prime}(t_i)\left[H^{*\prime}(t_i)\,P^*(t_i, t_{i-1})\,H^{*\prime}(t_i) + R^*(t_i)\right]^{-1} \tag{3-99}$$

$$P^*(t_i) = \left[I - K^*(t_i)\,H^*(t_i)\right]P^*(t_i, t_{i-1}) \tag{3-100}$$

Where:

$$\emptyset^*(t_i, t_{i-1}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ g(t_i - t_{i-1}) & 1 & \emptyset_{23} & \emptyset_{24} \\ 0 & 0 & \emptyset_{33} & 0 \\ 0 & 0 & 0 & \emptyset_{44} \end{bmatrix}$$

$$G^*(t_i) = \begin{bmatrix} 0 & 0 \\ G_{31} & G_{32} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad Q^*(t_i) = \sigma_V^2\left[1 - e^{-2(t_i - t_{i-1})/\tau}\right] I$$

$$\hat{x}^1(t_i) = \begin{bmatrix} \hat{\beta}(t_i) \\ \hat{p}_1(t_i) \\ \hat{n}_1(t_i) \\ \hat{n}_2(t_i) \end{bmatrix} \qquad x_2(t_i) = \begin{bmatrix} \hat{\gamma}(t_i) \\ -\hat{p}_2(t_i) \\ \hat{n}_3(t_i) \\ \hat{n}_4(t_i) \end{bmatrix}$$

$$H^*(t_i) = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \qquad R^*(t_i) = \Delta^2/12$$

BIAS ERRORS IN THE SUBOPTIMAL FILTER

The three Euler angles describing the misalignment of the inertial reference CS vary with time. This is caused by the earth's rate crosscoupling effect. This effect is neglected when the simplifying approximations are made and thereby causes a bias to occur in the estimates made by the si. plified filter.

The quantities one wishes to estimate are the Euler angles at the final time, t_f, since this is when the CTMC will be "torqued" to drive these angles to zero. The relation between the Euler angles at time t and those at time t_f is given by:

$$\theta(t) = \emptyset_{11}(t, t_f)\,\theta(t_f) \tag{3-101}$$

The accelerometer measurements $p(t_i)$ are thus related to $\theta(t_f)$ by:

$$p(t_i) = \int_{t_o}^{t_i} \emptyset_{11}(\lambda, t_f)\, d\lambda \;\; \theta(t_f) + V(t_i) - V(t_o) \tag{3-102}$$

It is assumed in the simplified dynamics that:

$$\emptyset_{11}(t_1, t_2) = I \tag{3-103}$$

The assumed measurement is that given by:

$$p^*(t_i) = B \int_{t_o}^{t_i} I \, d\lambda \;\theta(t_f) + V(t_i) - V(t_o) \tag{3-104}$$

Thus, the actual measurement differs from the assumed measurement by:

$$b(t_i) = p(t_i) - p^*(t_i) = B \int_{t_o}^{t_i} \left[\emptyset_{11}(\lambda, t_f) - I\right] d\lambda \;\theta(t_f) \tag{3-105}$$

This measurement bias causes a bias in the estimates of $\beta(t_f)$ and $\gamma(t_f)$.

A correction for this bias can be made. At time t_f, there are three quantities available from which we can estimate α, β and γ. The first quantity is the theodelite measurement. This is given by

$$m_1 = \alpha(t_f) + C_1\,\beta(t_f) + C_2\,\gamma(t_f) \tag{3-106}$$

The other two quantities are the biased estimates of β (t_f) and $\gamma(t_f)$. They are given by:

$$m_2 = \beta(t_i) \tag{3-107}$$

$$m_3 = \gamma(t_i) \tag{3-108}$$

Using these three "measurements" we can obtain a minimum variance estimate of θ (t_f) which will be unbiased. This estimate is related to the measurements by

$$\hat{\theta}(t_f) = A\, m \tag{3-109}$$

Where:

$$m = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}$$

To calculate A, one employs the principal of orthogonality which states that if θ (t_f) is to be a minimum variance estimate of θ (t_f), then

$$E\left\{\left[\hat{\theta}(t_f) - \theta(t_f)\right] m'\right\} = 0 \tag{3-110}$$

Substituting 3-109 into 3-110 and solving for A gives

$$A = P_{\theta m}\; P_{mm}^{-1} \tag{3-111}$$

Where

$$P_{\theta m} = E\left[\theta(t_f) m'\right] \tag{3-112}$$

$$P_{mm} = E\left[mm'\right] \tag{3-113}$$

The covariance matrix of the crosscoupling estimate is given by

$$P_{\hat{\theta}\hat{\theta}} = P_{\theta\theta} - AP'_{\theta m} \tag{3-114}$$

The correction matrix given by equation 3-111 provides the optimum correction in a least squares sense. However, a simpler correction matrix that produces adequate accuracy is derived in Section III-F. Since this correction matrix is relatively easy to calculate, it has been used for both types of filters.

D. LEAST SQUARES CURVE FIT FILTER

Introduction

This section presents the results of an investigation of the use of a least squares curve fit filter for levelling a strap-down system in a swaying missile. The objective of this investigation was to determine the simplest filter which would produce acceptable accuracy with a final data gathering period of less than 100 seconds. The simplest filter, of course, would be one in which the accelerometer pulses were merely counted and divided by time. However, a simple calculation shows that the error using this technique at the end of 100 seconds of time is much too large. The next simplest candidate filter is one in which the accelerometer output is assumed to be a power series in time. The accelerometer output data can then be fit to this power series in a manner which minimizes the square of the error. No a-priori knowledge of the form of any noise which corrupts the accelerometer output is assumed in deriving the estimated coefficients of the power series. However, it can be shown that the coefficients are the same as would be derived using a maximum likelihood approach with a white noise input.

Least Squares Curve Fit Equations

The sum of the accelerometer outputs at time t is given by

$$V = a_o + a_1 t + a_2 t^2 + \ldots\ldots a_n t^n = mA$$

where

$$m = [\, 1,\ t,\ t^2,\ \ldots\ldots\ t^n \,]$$

$$A = \begin{bmatrix} a_o \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$$

Now if K data points are taken at times, t_i, the error in the ith data point is

$$e_i = V_i - m_i \hat{A}$$

where $\hat{A}$ is the estimate of A

Then the sum of the square of the errors, which is the function to be minimized is

$$e^2 = \sum_{i=1}^{K} (V_i - m_i \hat{A})^2$$

Differentiating with respect to A and setting equal to zero yields

$$\frac{\partial e^2}{\partial A} = -2 \sum_{i=1}^{K} m_i^T (V_i - m_i \hat{A}) = 0$$

or

$$\sum_{i=1}^{K} m_i^T (V_i - m_i \hat{A}) = 0 \tag{3-115}$$

Equation 3-115 represents a set of n + 1 equations of the form

$$\sum_{i=1}^{K} (V_i - m_i \hat{A}) = 0$$

$$\sum_{i=1}^{K'} t_i (V_i - m_i \hat{A}) = 0$$

$$\vdots$$

$$\sum_{i=1}^{K} t_i^n (V_i - m_i \hat{A}) = 0$$

The solution of equation 3-115 yields

$$\sum_{i=1}^{K} m_i^T m_i \hat{A} = \sum_{i=1}^{K} m_i^T v_i$$

$$\hat{A} = \left[\sum_{i=1}^{K} m_i^T m_i \right]^{-1} \sum_{i=1}^{K} m_i^T V_i$$

$$\begin{bmatrix} \hat{a}_o \\ \hat{a}_i \\ \hat{a}_2 \\ \hat{a}_n \end{bmatrix} = \begin{bmatrix} \Sigma i & \Sigma t_i & \Sigma t_i^2 \ldots \Sigma t_i^n \\ \Sigma t_i & \Sigma t_i^2 \ldots & \vdots \\ \Sigma t_i^2 & & \vdots \\ \Sigma t_i^n \ldots\ldots & & \vdots \end{bmatrix}^{-1} \begin{bmatrix} \Sigma v_i \\ \Sigma t_i V_i \\ \Sigma t_i^2 V_i \\ \Sigma t_i^n V_i \end{bmatrix} \tag{3-116}$$

Equation 3-116 is the basic least squares curve fit equation used in this technique.

Physical Model

The acceleration sensed by an accelerometer at a small angle θ from the horizontal and with zero sway velocity is

$$a = \theta g$$

Consider the coordinate system of Figure 20. In general

$$\left.\begin{aligned}
\theta &= \theta_o + \int \dot{\theta}\, dt \\
\dot{\theta} &= \dot{\theta}_d + \epsilon W_{e2} = \delta We_1 \\
\epsilon &= \epsilon_o + \int \dot{\epsilon}\, dt \\
\dot{\epsilon} &= \dot{\epsilon}_d + \delta W_{e1} - \theta W_{e2} \\
\delta &= \delta_o + \int \dot{\delta}\, dt \\
\dot{\delta} &= \dot{\delta}_d + \theta We_1 - \epsilon We_2
\end{aligned}\right\} \qquad (3\text{-}117)$$

Here ϵ, δ, θ are the angular misalignments, the sub 0 quantities are initial misalignments, and the sub d quantities are due to gyro drift.

For simplicity in determining the form of the accelerometer output, assume that the 3 axis is North, then $W_{e2} = 0$ and

$$\dot{\theta} = \dot{\theta}_d - \delta\, We_1$$

Ignoring second order effects

$$\delta = \delta_o + \theta_o W_{e1}\, t$$

and

$$\dot{\theta} = \dot{\theta}_d - \delta_o We_1 - \theta_o W_{e\,1}^2\, t$$

$$\theta = \theta_o + (\dot{\theta}_d - \delta_o We_1)\, t - 1/2\, \theta_o\, W^2_{e1}\, t^2$$

The sum of the pulses is the integral of a and is given by

$$V = g\, [\theta_o t + 1/2\, (\dot{\theta}_d - \delta_o We_1) t^2 - 1/6\, \theta_o We^2_1\, t^3] \qquad (3\text{-}118)$$

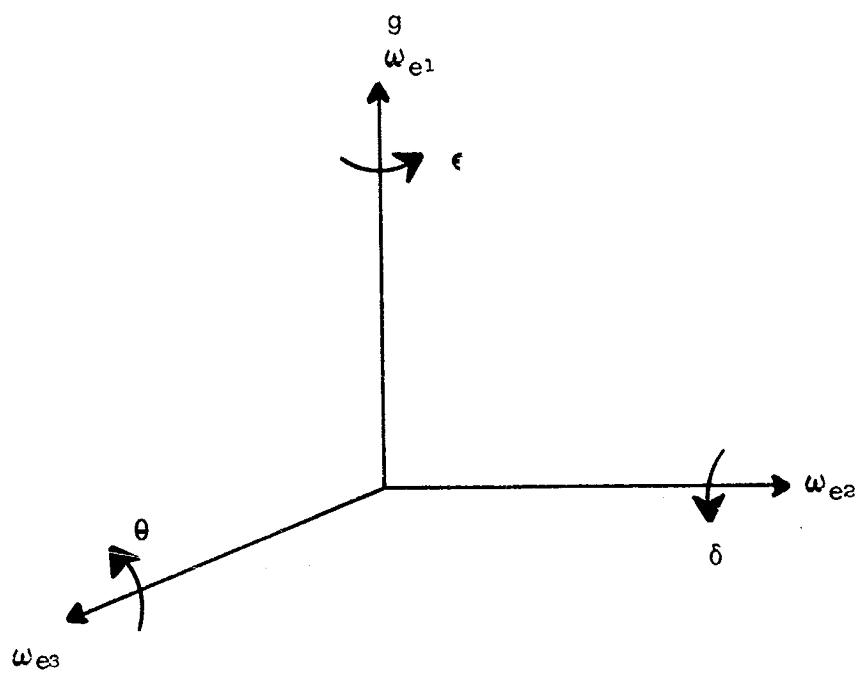


Figure 20.

Now if sway velocity is considered including the fact that the velocity output of the accelerometer has an average value, 3-118 can be written

$$V = V_o + g\ [\theta_o t + 1/2\ (\dot{\theta}_d - \delta W_{e1})t^2 - 1/6\ \theta_o W_{e1}{}^2 t^3] \tag{3-119}$$

If V is fit to a curve of the form $V = a_o + a_1 t + a_2 t^2 + \ldots.$, the initial misalignment angle, θ_o, and rate of change angle, $\dot{\theta}$, can be determined as

$$\theta_o = \frac{a1}{g} \tag{3-120}$$

$$\dot{\theta} = \frac{2\,a_2}{g} \tag{3-121}$$

Effect of Sinusoidal Noise at a Single Frequency

If the noise is assumed to be a sinusoid at a single frequency and data is taken at fixed time increments, a simple paper analysis can be performed to determine the errors as a function of noise magnitude, noise frequency, time of filtering and order of fit.

Assuming the data is taken at times δt, $2\delta t$, ...$K\delta t$ the various terms in equation 3-116 become

$$\Sigma i = K$$

$$\Sigma t_i = \delta t\,[1 + 2 + 3 + \ldots] = \delta t\ \frac{K\,(K+1)}{2}$$

$$\Sigma t^2{}_i = \delta t^2\,[1 + 4 + 9 + \ldots] = \delta t^2\ \frac{K\,(K+1)\ (2K+1)}{6}$$

etc.

Now if K >>1 (many data points are taken) at time $T = K\delta t$

$$\Sigma i = \frac{1}{\delta t}\ T$$

$$\Sigma t_i = \delta t\ K^2/2 = \frac{1\ T^2/2}{\delta t}$$

$$\Sigma t^2{}_i = \frac{1}{\delta t}\ T^3/3$$

etc.

or

$$\Sigma i = \frac{1}{\delta t} \int_0^T dt$$

$$\Sigma t_i = \frac{1}{\delta t} \int_0^T t \, dt$$

$$\Sigma t_i^2 = \frac{1}{\delta t} \int_0^T t^2 \, dt$$

Similarly

$$\Sigma V_i = \frac{1}{\delta t} \int V_i \, dt$$

$$\Sigma t V_i = \frac{1}{\delta t} \int t \, V_i \, dt$$

etc.

And 3-116 becomes

$$\begin{bmatrix} \hat{a}_o \\ \hat{a}_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \hat{a}_n \end{bmatrix} = \begin{bmatrix} T & T^2/2 \ldots \\ T^2/2 & T^3/3 \ldots \\ \cdot & \\ \cdot & \\ \cdot & \\ \cdot & \\ \cdot & \end{bmatrix}^{-1} \begin{bmatrix} \int_0^T V_i \, dt \\ \int_0^T t \, V_i \, dt \\ \int_0^T t^2 \, V_i \, dt \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \qquad (3\text{-}122)$$

Now suppose the data is given by $\Sigma V_i = a_o + a_1 t + a_2 t^2 + \ldots a_n t^n + \text{Noise}$ (3-123)

If the data is now fit to

$$\Sigma V_i^1 = \hat{a}_o + \hat{a}_1 t + \hat{a}_2 t^2 + \ldots . \hat{a}_m t^m \quad (m < n)$$

Equation 3-123 can be written

$$\Sigma V_i = \Sigma V_i^1 + a_{m+1}t^{m+1} + \ldots a_n t^n + \text{Noise} \tag{3-124}$$

If ΣV_i consisted only of ΣV_i^1 the estimates of the coefficients, $\hat{a}_o, \hat{a}_1 \ldots\ldots \hat{a}_m$ would be correct. Errors in the coefficients are caused by the other terms in 3-124 .

Letting $\delta V_i = a_{m+1}t^{m+1} + \ldots a_n t^n + \text{Noise}$.

The errors in the coefficients are given by

$$\begin{bmatrix} \delta a_0 \\ \\ \delta a_1 \\ \\ \delta a_2 \\ \vdots \\ \delta a_m \end{bmatrix} = \begin{bmatrix} T & T^2/2 & \ldots \\ \\ T^2/2 & T^3/3 & \ldots \\ \cdot & \cdot & \\ \cdot & \cdot & \\ \cdot & \cdot & \end{bmatrix}^{-1} \begin{bmatrix} \int_0^T \delta V_i \, dt \\ \int_0^T t \, \delta V_i \, dt \\ \int_0^T t^2 \, \delta V_i \, dt \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \tag{3-125}$$

First consider fitting the data to a quadratic function $(a_o+a_1t+a_2t^2)$ with noise given by

$$\delta V_i = C [\sin(Wt+\emptyset) - \sin\emptyset]$$

$$\int_0^T \delta V_i \, dt = \frac{C}{W} [\cos\emptyset - \cos(WT+\emptyset)] - CT\sin\emptyset \tag{3-126}$$

$$\int_0^T tV_i dt = \frac{C}{W} [\frac{1}{W} \sin(WT+\emptyset - \frac{1}{W}\sin\emptyset - T\cos(Wt+\emptyset)] - (CT^2/2)\sin\emptyset$$

$$\int tV_i dt \approx -\frac{C}{W} [T\cos(WT+\emptyset)] - (CT^2/2)\sin\emptyset \text{ for } WT >> 1 \tag{3-127}$$

$$\int t^2 V_i At = \frac{C}{W} \left[\frac{2T}{W}\sin(WT+\emptyset) + \frac{2}{W2}\cos(WT+\emptyset) - \frac{2}{W2}\cos\emptyset - T^2\cos(WT+\emptyset)\right] - \frac{CT3}{3}\sin\emptyset$$

$$\int t^2 V_i dt \approx \frac{C}{W} [-T^2\cos(WT+\emptyset)] - \frac{CT^3}{3}\sin\emptyset \text{ for } WT >> 1 \tag{3-128}$$

For a quadratic fit equation 3-125 becomes

$$\begin{bmatrix} \delta a_0 \\ \delta a_1 \\ \delta a_2 \end{bmatrix} = \frac{1}{T^5} \begin{bmatrix} 9T^4 & -36T^3 & 30T^2 \\ -36T^3 & 192T^2 & -180T \\ 30T^2 & -180T & 180 \end{bmatrix} \begin{bmatrix} \int_0^T \delta V_i \, dt \\ \int_0^T t\delta V_i \, dt \\ \int_0^T t^2 \delta V_i \, dt \end{bmatrix} \tag{3-129}$$

Using 3-126, 3-127, and 3-128

$$\delta a_1 = \frac{12C}{WT^2} \left[2 \cos (Wt + \emptyset) - 3 \cos \emptyset \right] \tag{3-130}$$

The rms value over all $\emptyset$ and WT is

$$\delta a_{1rms} = \frac{30.5\ C}{WT^2} \tag{3-131}$$

Using 3-120

$$\delta \theta_{0\ rms} = \frac{30.5\ C}{gWT^2} \tag{3-132}$$

$$\delta a_2 = \frac{30C}{WT^3} \left[\cos \emptyset - \cos (WT + \emptyset) \right]$$

The rms value over all $\emptyset$ and WT is

$$\delta a_{2rms} = \frac{30C}{WT^3}$$

Using 3-121

$$\delta \dot{\theta}_{rms} = \frac{60C}{gWT^3} \tag{3-133}$$

If the angle is estimated at time T, the error in the estimate is given by

$$\delta \theta_T = \frac{\delta a_1 + 2\delta a_2 T}{g} = - \frac{12C}{gWT^2} \left[2 \cos \emptyset - 3 \cos (WT + \emptyset) \right]$$

$$\delta \theta_{Trms} = \frac{30.5C}{gWT^2}$$

Using 3-132, 3-133, and 3-134 with $\frac{C}{W}$ = 1/2 meter and T = 100 seconds, the following errors can be calculated

$$\left.\begin{aligned} \delta\theta_{0rms} &\approx 30 \text{ arc seconds} \\ \delta\dot{\theta}_{rms} &\approx 0.6 \text{ degrees/hour} \\ \delta\theta_{Trms} &\approx 30 \text{ arc seconds} \end{aligned}\right\} \tag{3-134}$$

If an accuracy of 20 arc-seconds is desired, it is obvious that a quadratic curve fit is not acceptable. It is also obvious that no benefit is gained by attempting to estimate the drift rate and using this to update the angle estimate. Therefore, it was decided to try a linear curve fit.

For a linear fit, equation 3-125 becomes

$$\begin{bmatrix} \delta a_0 \\ \\ \delta a_1 \end{bmatrix} = \frac{1}{T^3} \begin{bmatrix} 4T^2 & -6T \\ \\ -6T & 12 \end{bmatrix} \begin{bmatrix} \int_0^T \delta V_i \, dt \\ \\ \int_0^T t\delta V_i dt \end{bmatrix} \tag{3-135}$$

Using 3-126, 3-127, and 3-128

$$\delta a_1 = \frac{6C}{WT^2} \left[\cos (WT + \phi) - \cos \phi \right]$$

$$\delta a_{1rms} = \frac{6C}{WT^2}$$

For C/W = 1/2 meter T = 100 seconds

$$\delta\theta_{0rms} = \frac{6C}{gWT^2} = 6 \text{ seconds of arc}$$

This error due to sinusoidal noise is perfectly acceptable. However, from equation 3-119 it is known that there are quadratic, cubic and higher powered second order terms in the data. The effects of these terms must be investigated.

Let $\delta V_i = 1/2g\,(\dot{\theta}_d - \delta_0 W_{e1})\,t^2$

Then using 3-135

$$\delta a_1 = \frac{g}{2} (\dot{\theta}_d - \delta_0 W_1)\, T$$

$$\delta\theta_0 = 1/2\, (\dot{\theta}_d - \delta_0 W)\, T$$

It should be noted that the error at time T in the estimate of the initial angle is one-half the actual change in the angle from time zero to time T due to drift rate and earth's rate coupling. Since the desired answer is the angle at time T and not the initial angle, the net result is an error in the angle at time, T, which is equal to 1/2 the change in the angle due to $\dot{\theta}$ over the time interval T. This error can be significant if there are large drift rates or initial angle errors.

Now consider the error due to a cubic term in the data.

Let $\delta V_i = 1/6\, g\, \theta_0\, W_1{}^2\, t^3$

Then using 3-135

$$\delta a_1 = \frac{0.9}{6}\; g\, \theta_o\, W_1^2\; T^2$$

$$\delta \theta_0 = 0.15\; \theta_0\, W_1^2\; T^2$$

Assuming $\theta_0 = .01$

W_{e1} = full earth's rate

T = 100 seconds

$\delta \theta_0 \approx 0.02$ seconds of arc

This error is negligible and higher powered terms in the data will have even less effect.

From the results of this paper analysis, several conclusions can be drawn.

First, a quadratic fit does not produce acceptable accuracy.

Second, if a linear fit is used, there may be a problem with drift rate or earth's rate coupling. For example, if there is an initial angle error of .02 radians, an earth's rate coupling error of 0.3 seconds/second can exist. At the end of 100 seconds, the change in angle is 30 seconds of arc and only one half the change in angle is compensated for by the error in the estimate of the initial angle. This problem can be eliminated in several ways that are discussed as follows:

a) Two iterations can be used, the first iteration would last for 25-30 seconds and reduce angle errors to about 0.5 milliradians by updating the direction cosine matrix in the CTMC. This procedure essentially eliminates the earth's rate coupling. If gyro drift is truly negligible, a second iteration of 100 seconds will yield an angle estimate with a 1 σ accuracy of about 6 seconds of arc.

b) The data could be corrected for the initial misalignment by using equations similar to 3-117 and then refit. This procedure is more complex than the two iteration with updating procedure.

c) The simplest, most straightforward approach is to use the technique described in the Crosscoupling Corrections Section of this report. This is the recommended approach.

Computer Analysis

To confirm the results of the paper analysis with more representative noise models, a computer analysis was undertaken. Figure 21 shows the computer program in block diagram form. A polynominal in time is generated and added to noise to provide the accelerometer output. The noise may be generated in two ways. Either narrow band random noise with an autocorrelation function $\sigma_V^2 e^{-a\tau} \cos W_0\tau$ can be used or the noise can be represented as the sum of sinusoidal terms. The accelerometer output is quantized and fed into a least squares curve fit program to provide estimates of the polynominal coefficients.

Figures 22-27 show the results of several runs made with random noise inputs. All runs assume a quantization level of 0.02 m/second, a noise center frequency of 1.57 radians/second and sampling time of 1 second.

These curves show the envelope of points obtained with the individual prints marked except for those on the envelope. All the curves show the beating effect in the error which has been evident in the maximum likelihood and Kalman Filters with imperfect noise models and in general confirm the results of the paper analysis.

Figure 22 shows the results of a quadratic fit after about 100 seconds time with 20 meru drift. Rather large errors in the estimate of the initial angle are evident. Since the change in angle over 100 seconds time is +30 seconds of arc, the errors in the actual angle estimate are even larger.

Figure 23 shows the results of a linear fit after about 100 seconds time with 20 meru drift. The change in actual angle over 100 seconds time is again +30 seconds of arc. It is seen that the error made in the estimate of the initial angle compensates for about one half of this change as predicted by the paper analysis.

Figures 24 and 25 show the errors existing around 25 seconds with a linear fit for two different correlation times. These results confirm that an initial iteration will reduce angle errors to about 100 seconds of arc.

Figures 26 and 27 show the errors existing after 100 seconds with a linear fit and an initial condition of 0.0005 radians for two different correlation times. This initial condition would be typical for a second iteration if an iterative technique were used. However, it should be pointed out that the error is not a strong function of initial condition (see Table 2). Therefore the results are valid even if only one iteration is used. The rms angle error determined from these runs are shown in Table 1. The errors are determined at each sampling time from 98 to 102 seconds and over the whole time interval from 98 to 102 seconds. The earth's rate coupling error at this time is equivalent to 0.7 seconds of arc. The error due to gyro drift is given by 50 $\dot{\theta}_d$. $\dot{\theta}_d$ must be small enough to make this error acceptable for this technique to work.

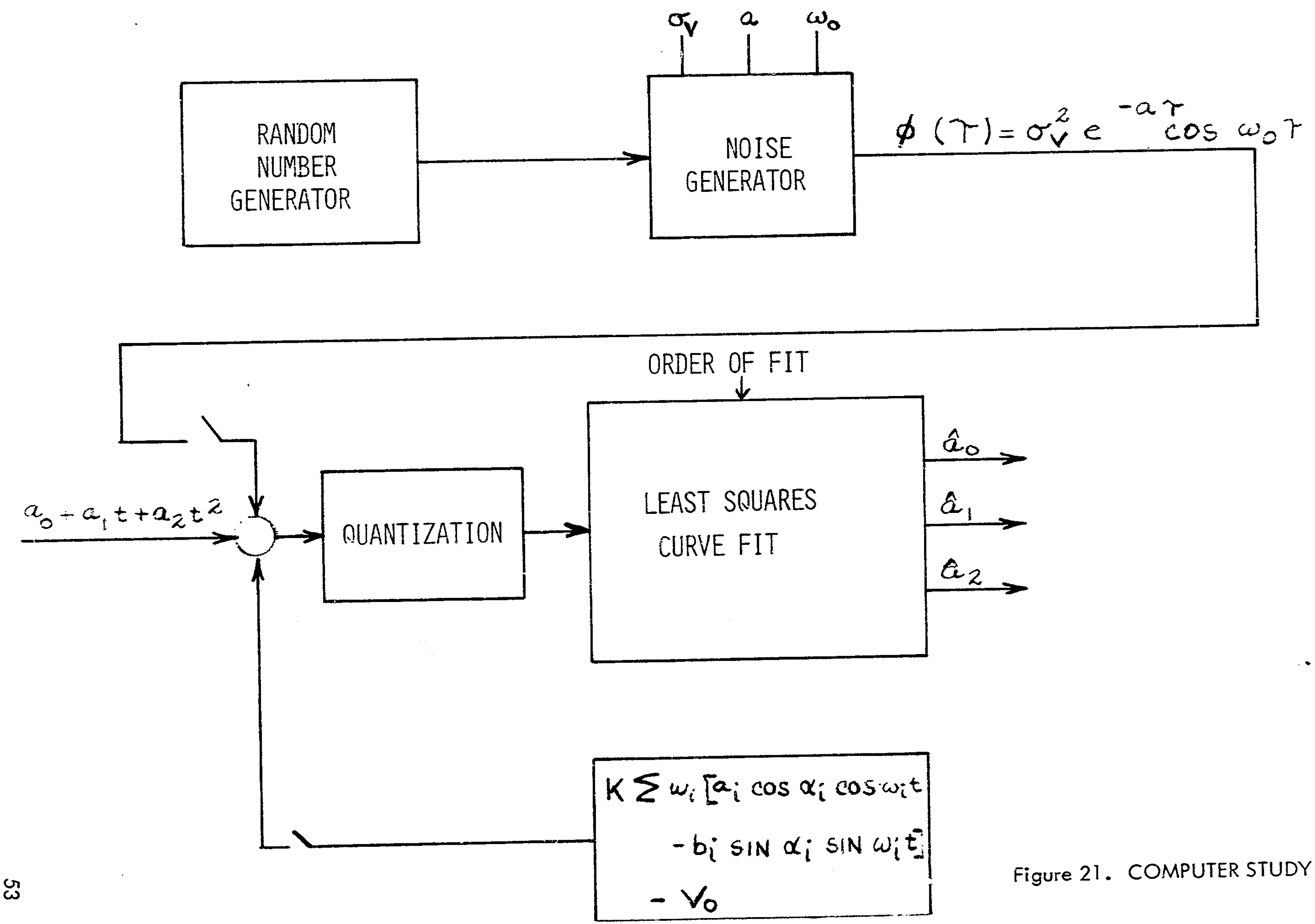


Figure 21. COMPUTER STUDY

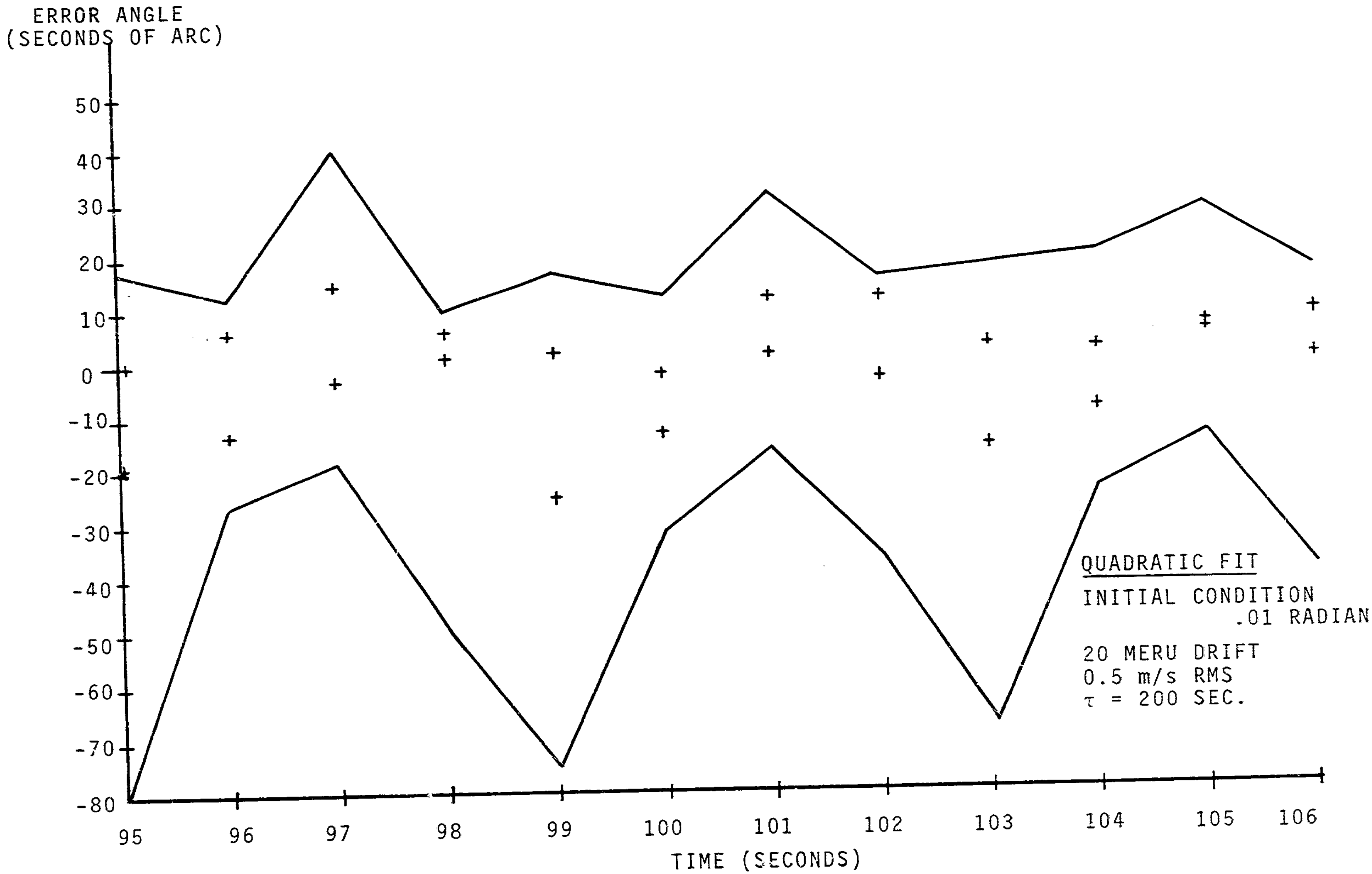


Figure 22. ERROR IN INITIAL ANGLE ESTIMATE

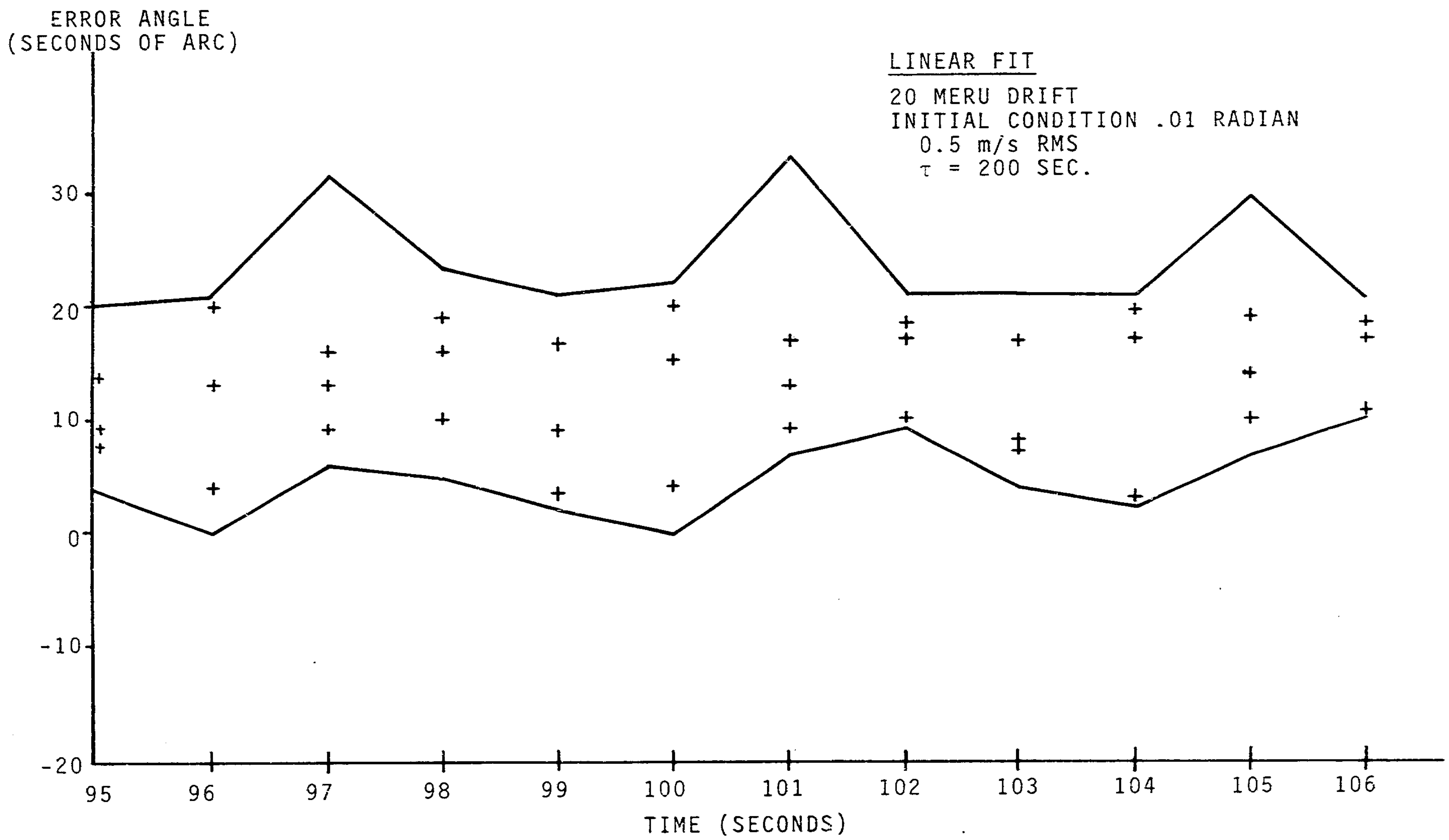


Figure 23. ERROR IN INITIAL ANGLE ESTIMATE

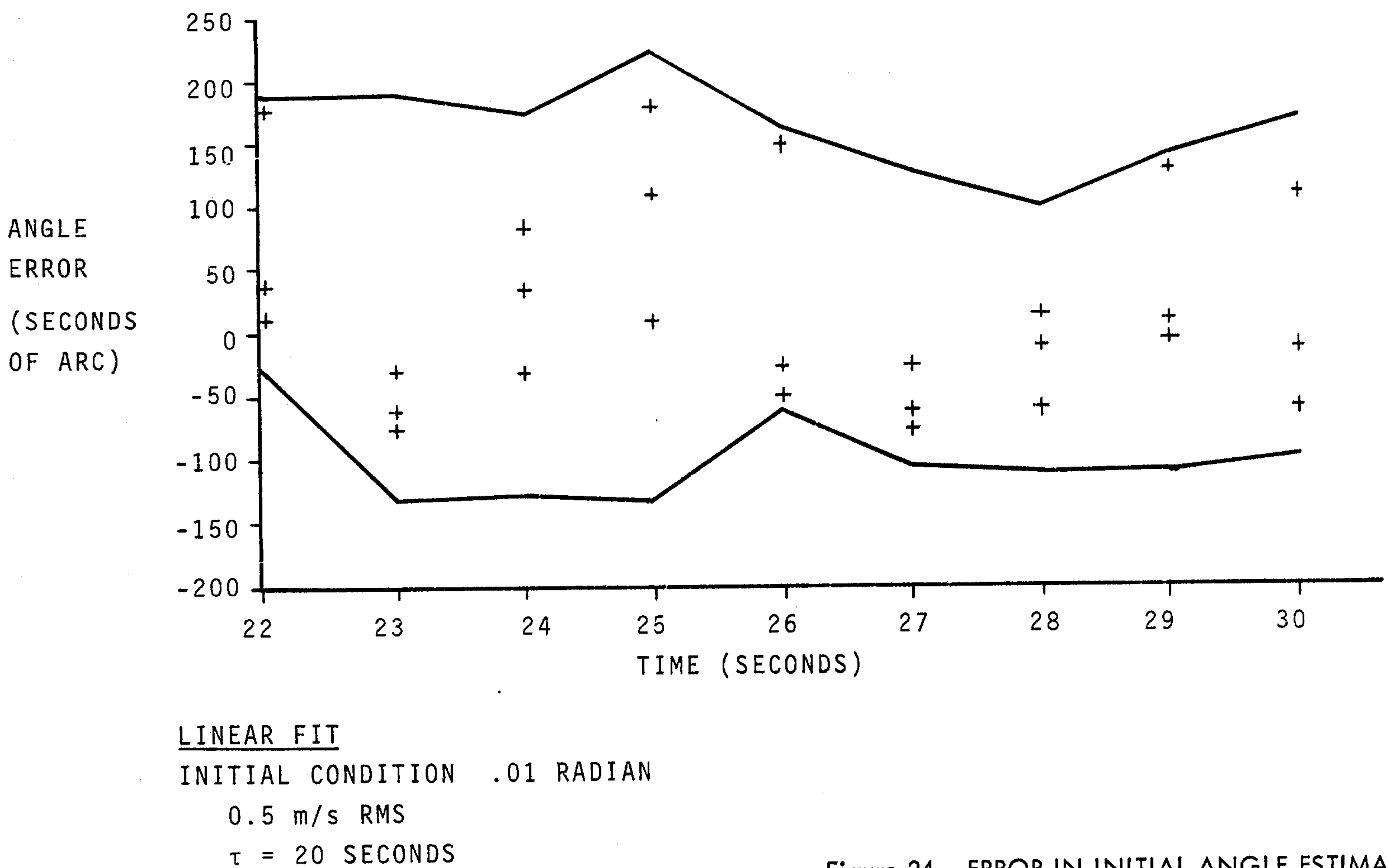


Figure 24. ERROR IN INITIAL ANGLE ESTIMATE

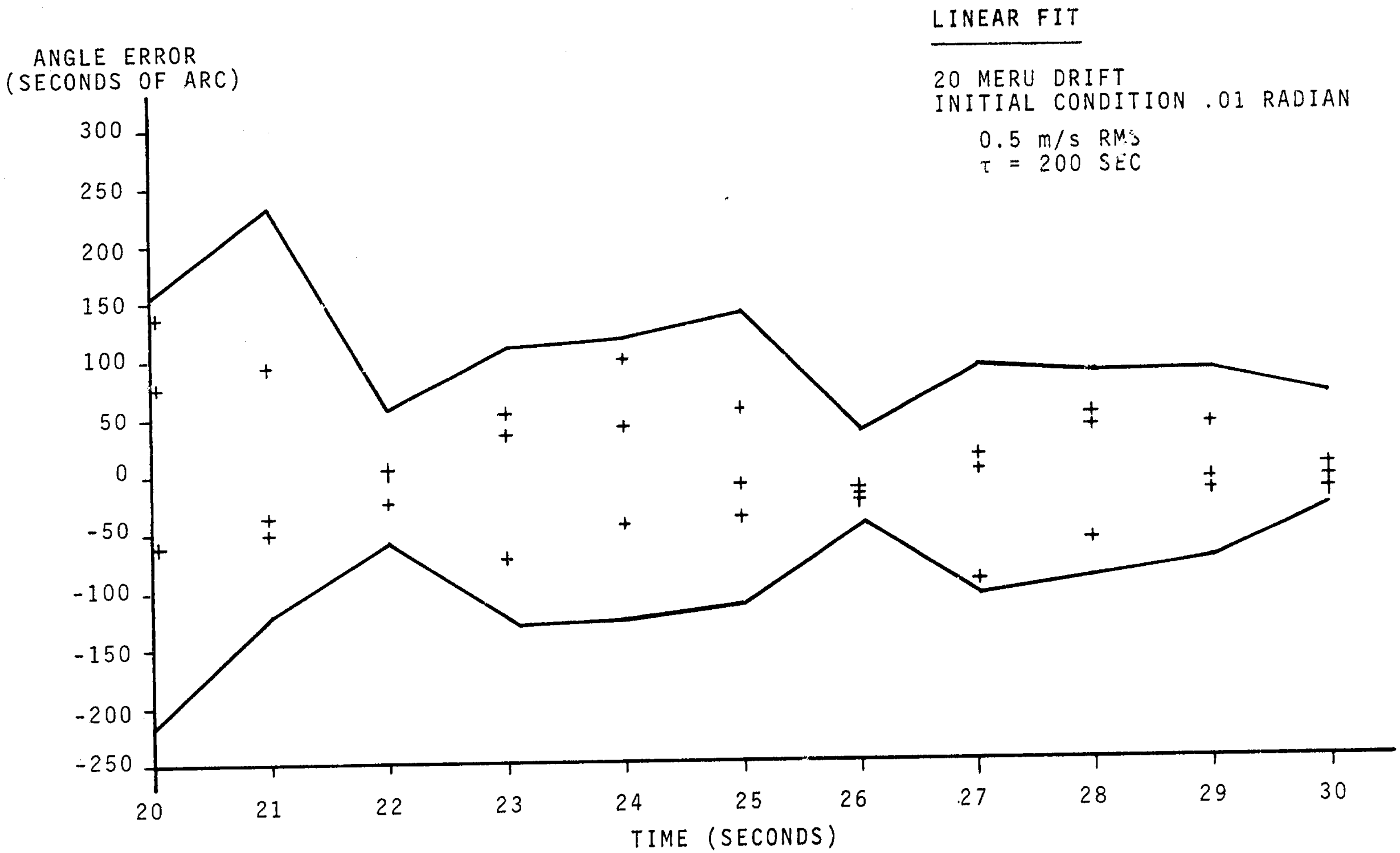


Figure 25. ERROR IN INITIAL ANGLE ESTIMATE

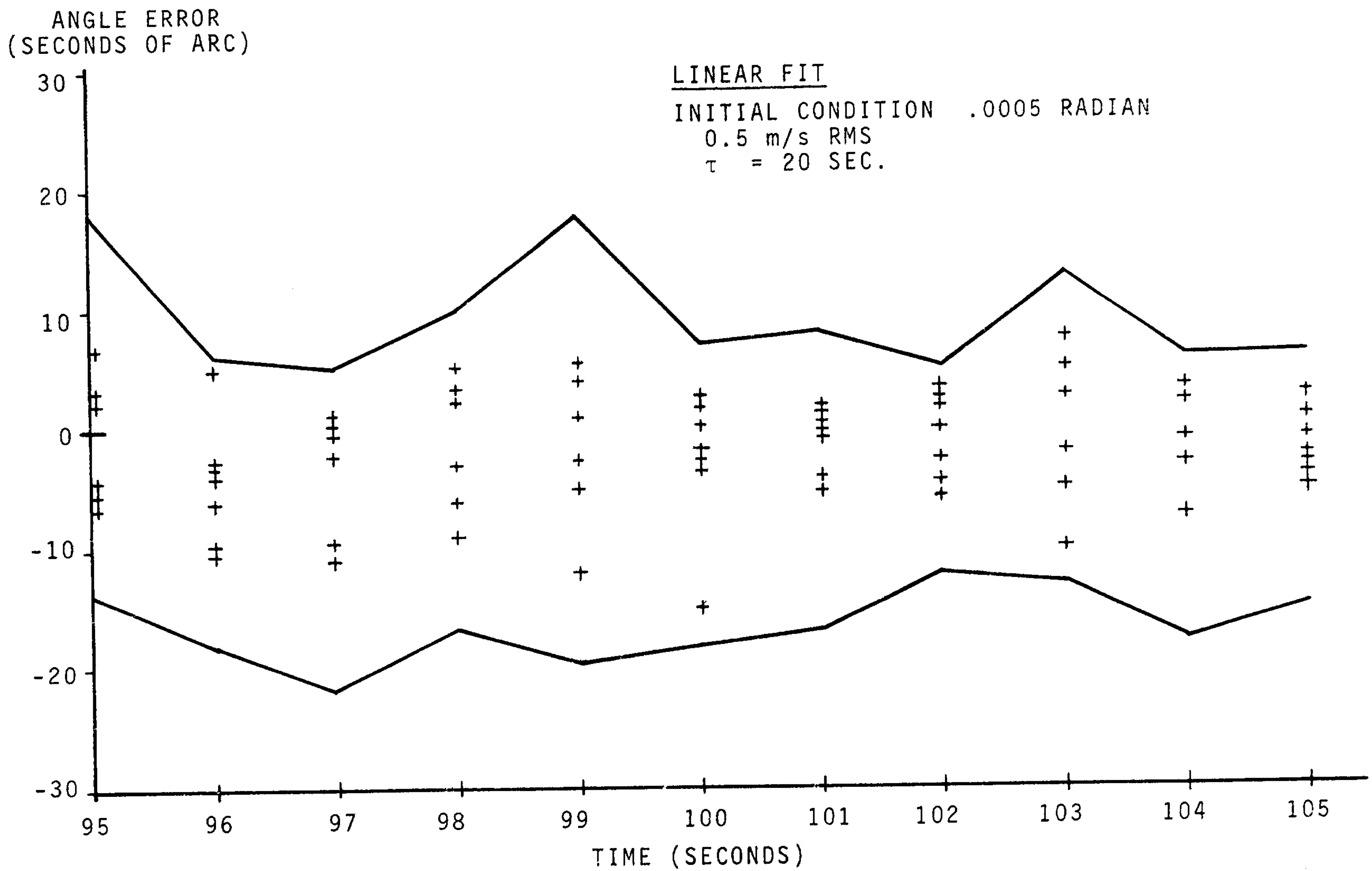
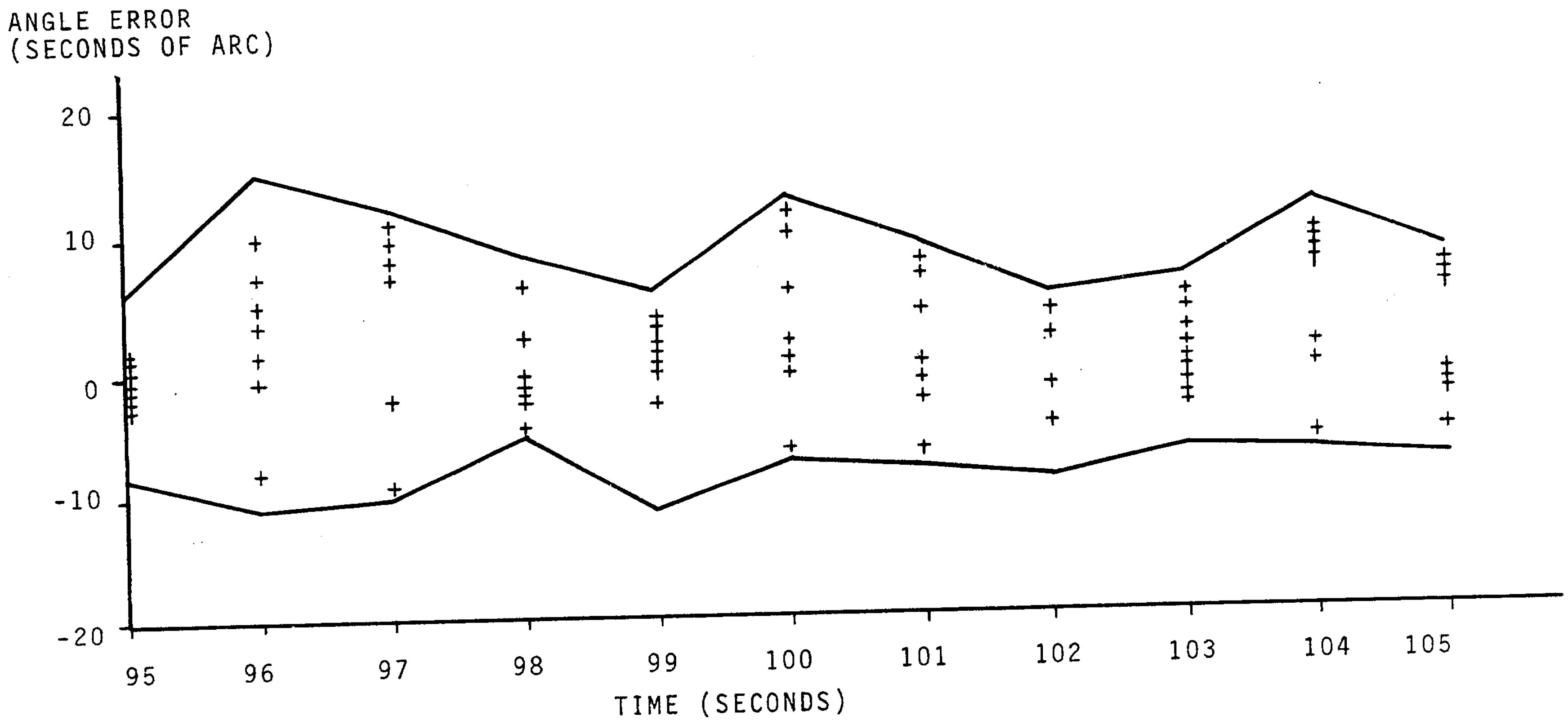


Figure 26. ERROR IN INITIAL ANGLE ESTIMATE



LINEAR FIT

INITIAL CONDITION .0005 RADIAN

0.5 m/s RMS

τ = 200 SEC.

Figure 27. ERROR IN INITIAL ANGLE ESTIMATE

Table 1
ANGULAR ERRORS AROUND 100 SECONDS OF TIME

Time (seconds)	RMS Angular Error (seconds of arc)	
	τ = 200 seconds	τ = 20 seconds
98	4.7	8.0
99	4.5	10.4
100	8.0	7.9
101	6.1	6.6
102	4.7	6.8
98-102	5.7	8.1

Table 2 indicates the errors obtained using noise that is the sum of sinusoids at four different frequencies. The noise is approximately equal to that used in NASA's CTMC Run SO4400060 shown in Table 2 also. Answers were obtained for two different initial conditions and indicate that the error is not a strong function of the initial condition. The errors are slightly less than those obtained using random noise.

Table 2
A. ERRORS USING SINUSOIDAL NOISE

Time (seconds)	Angular Error (seconds of arc)	
	0.01 radian initial angle	0.0005 radian initial angle
22	75	83
23	0	4
24	108	118
25	81	85
26	11	16
98	-5.5	-5
99	1	1
100	8	-7.5
101	-7.5	-6
102	0	0

Table 2 (Continued)

B. NOISE INPUT COMPARED TO CMTC RUN S4400060

Time (Seconds)	V_x (m/s) S04400060	V_x (m/s) OD
0	0	0
1	0.98	0.95
2	1.21	1.32
3	0.19	0.23
4	0.59	0.83
5	1.22	1.10
6	0.72	0.61

Preconditioning of Data

An analysis of the effects of preconditioning the data by integrating before curve fitting has been performed assuming sinusoidal noise at a single frequency. Noise is assumed to be

$$V_i = C\,[\sin(Wt + \phi) - \sin\phi]$$

The analysis follows that used in the effect of sinusoidal noise at a single frequency, covered earlier in this section. The integrated data is fit to a quadratic and the quadratic coefficient is the one of interest. Results show that

$$\delta a_{2rms} = \frac{30C}{W^2T^3}$$

$$\delta\theta_{0rms} = \frac{60C}{gW^2T^3}$$

Using $\frac{C}{W}$ = 1/2 meter W = 1.5 radians/second and T = 100 seconds yields

$$\delta\theta_{0rms} \approx 0.4 \text{ seconds of arc}$$

This error is significantly smaller than that obtained when the data is not integrated before curve fitting. The validity of this result with more representative noise was tested by modifying the computer simulation shown in Figure 21 so that the generated data plus noise is integrated being quantized and curve fit. The integration is performed by the simple summing procedure shown below.

$$\int_0^{t=K\Delta t} V\,dt = \sum_{i=0}^{K-1} V_i + V_K$$

A quadratic fit is used and the quadratic coefficient is the one of interest.

The results using random noise with an autocorrelation function of the form $\phi(t) = \sigma_v^2 e^{-a\tau} \cos \omega_0 \tau$ are shown in Figure 13 of the results section as a function of the correlation time, τ, of the noise. The error plotted is the rms error over the time period from 98 to 102 seconds from the start of the second iteration. Figure 13 also shows a comparison of the results obtained using non-integrated and integrated data. The non-integrated data is fit to a linear curve as described previously. For noise correlation times greater than about 20 seconds, the use of integrated data produces better results. For shorter correlation times, the use of integrated data produces worse results. The asymptotes on the figure are the errors when the noise is assumed to be a sine of wave of a single frequency ($\tau=\infty$).

The results using a noise input approximately equal to that used in NASA's CTMC run S04400060 are tabulated in Table 3 for a 0.01 radian initial angle.

Table 3

Time (Seconds	Angular Error (Seconds of arc)	
	Integrated Data	Non-Integrated Data
22	-7	75
23	+25	0
24	+ 1	108
25	- 5	81
26	+11	11
98	-0.3	-5.5
99	0.2	1
100	-0.2	8
101	-0.4	-7.5
102	+0.1	0

The noise in this case consists of a sum of sine waves and the results indicate a significant improvement in accuracy when the data is integrated.

However, it must be recognized that the noise will at least contain a random component with a correlation time which is unknown at this time. Since short correlation times can result in a severe degradation of performance when using integrated data, the preconditioned data approach was not pursued further.

Conclusions

1) A least squares curve fit technique can be used to produce accuracies better than 10 seconds of arc, 1σ. There are two alternate ways in which it can be mechanized. The first of these uses the technique described in section of this report to eliminate earth's rate couplings and requires a data gathering period of 100 seconds. This is the recommended approach. An alternate method requires two iterations with data gathering periods of 25 and 100 seconds respectively. This method requires that the CTMC be updated at the end of the first iteration to remove the earth's rate couplings.

2) An error equal to one half the angle change due to gyro drift over a 100 second data gathering interval exists. Gyro drift must be small enough to make this error acceptable.

3) More time is required to reach a given accuracy with this technique than with a Kalman filter.

4) The equations to be mechanized are simpler than for the Kalman filter. The equation for a linear least squares fit is

$$\begin{bmatrix} \hat{a}_0 \\ \hat{a}_1 \end{bmatrix} = \begin{bmatrix} \Sigma i & \Sigma t_i \\ \Sigma t_i & \Sigma t_i^2 \end{bmatrix}^{-1} \begin{bmatrix} \Sigma V_i \\ \Sigma t_i V_i \end{bmatrix}$$

Since a constant sampling interval may be used, the coefficients of the matrix inverse may be precalculated and stored.

The computation problem is then reduced to solving the equation

$$\hat{\theta} = \frac{\hat{a}_1}{g} = b \Sigma V_i + c\, \Sigma t_i V_i$$

Here b and c are precalculated constants and the equations for calculating them are given in Appendix C.

E. MAXIMUM LIKELIHOOD FILTER

The maximum likelihood filter will be applied to determine the parameters of a deterministic function when measurements of this function are corrupted by noise. For many cases this filter yields the same results as a Kalman filter and these similarities will be discussed later. Out total objective is to investigate errors in the filtering process when the assumed or modeled noise is inconsistent with the true noise.

DYNAMIC MODEL

We consider one axis of a platform system having an initial misererection β and a constant drift rate d. Earth's motion is not considered. Then the misererection angle θ is given by

$$\theta = \beta + dt \tag{3-136}$$

An accelerometer mounted at this angle would sense gravity in the amount

$$a = g\theta = g\beta + gdt \tag{3-137}$$

The velocity output would then be

$$v = g\beta t + \frac{gdt^2}{2} \tag{3-138}$$

The initial misalignment and drift are random parameters, or rather random constants. Thus the velocity (equation 3-138) is deterministic and the filtering of data to determine the coefficients of the linear and quadratic time terms will allow the initial misalignment and drift to be found.

A sequence of velocity measurements are made using an integrating accelerometer. The actual measurement is denoted by Z. Due principally to missile sway, the measurement is noisy. Thus, a general measurement is

$$Z_i = A_1 t_i + A_2 t_i^2 + E_i \tag{3-139}$$

where A_1 and A_2 are the coefficients of the deterministic terms and E is the noise input. For a sequence of n measurements

$$\begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_n \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{bmatrix} - \begin{bmatrix} t_1 & t_1^2 \\ t_2 & t_2^2 \\ \vdots & \vdots \\ t_n & t_n^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \tag{3-140a}$$

or

$$E = Z - UA \tag{3-140b}$$

We consider the error is represented by Gaussian noise with covariance matrix S, i.e.

$$S = < E E^T > \tag{3-141}$$

Thus, the joint probability density function for the errors is

$$p(Z) = \frac{1}{(2\pi)^{n/2}/S/^{1/2}} e^{-1/2 \, (E^T S^{-1} E)}$$

$$= \frac{1}{(2\pi)^{n/2}/S/^{1/2}} e^{-1/2 \, (Z-UA)^T S^{-1} (Z-UA)} \tag{3-142}$$

Maximum Likelihood Estimate

This estimate is found by determining the parameters A_1 and A_2 which maximize p(Z), equation 3-142 , for the actual set of observations. Several significant points must be noted-------- 1) we assume now no apriori knowledge of the coefficients, i.e. initial variances of A_1 and A_2 are infinite; 2) the parameter A_1 and A_2 are independent; this would not necessarily be true if earth's rotation were considered.

Thus, maximization of p(Z) can be done relative to A_1 and A_2 directly. Let

$$L(A) = (Z-UA)^T S^{-1} (Z-UA) \tag{3-143}$$

We maximize the taking the gradient of L(A)

$$\text{grad}_A \, L(A) = -2U^T S^{-1} (Z-UA) = 0$$

or

$$U^T S^{-1} UA = U^T S^{-1} Z$$

So that the estimated value of the coefficients is

$$\hat{A} = (U^T S^{-1} U)^{-1} U^T S^{-1} Z \tag{3-144}$$

This equation provides the weighting coefficients for determining A in terms of the measured velocities. To estimate the error in $\tilde{A}$ due to the noise, we realize that

$$Z = UA + E \tag{3-145}$$

where the true but unknown value of A is used.

Substituting for Z in equation 3-144 .

$$\tilde{A} = (U^T S^{-1} U)^{-1} (U^T S^{-1} U) A + (U^T S^{-1} U)^{-1} U^T S^{-1} E$$

or

$$\Delta A = \tilde{A} - A = (U^T S^{-1} U)^{-1} U^T S^{-1} E \tag{3-146}$$

We note that the estimate is unbiased since the expected value of E is zero. The variance of A is

$$P_A = \langle (\Delta A)(\Delta A)^T \rangle \tag{3-147}$$

In taking the transpose, we make use of the fact that S and $U^T S^{-1} U$ is symmetric. Thus,

$$P_A = \langle (U^T S^{-1} U)^{-1} U^T S^{-1} E E^T S^{-1} U (U^T S^{-1} U)^{-1} \rangle \tag{3-148}$$

We now consider that the true noise may have a covariance differing from the modeled noise, equation 3-141. Thus, let

$$S_T = \langle EE^T \rangle \tag{3-149}$$

Then the variance of the estimated coefficients is

$$P_A = (U^T S^{-1} U)^{-1} U^T S^{-1} S_T S^{-1} U (U^T S^{-1} U)^{-1} \tag{3-150a}$$

If the model is correct so that $S_T = S$, then

$$P_A = (U^T S^{-1} U)^{-1} \tag{3-150b}$$

These are the two fundamental equations used in all later calculations to compare the effects of inconsistencies between modeled and true noise.

Example and Use of A Priori Data

Velocity noise having an auto-correlation function

$$R(n\delta\tau) = .5^{-n\,\delta\tau/200}\cos 2\,n\delta\tau \quad \text{meter}^2/\text{sec}^2 \tag{3-151}$$

was used. Model noise matched true noise. Time spacing was $\delta\tau$ = 1 second and n = 25 points were used. Results for the variance are

$$P_A = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \doteq \begin{pmatrix} .281\times10^{-4} & -.110\times10^{-5} \\ -.110\times10^{-5} & .461\times10^{-7} \end{pmatrix}$$

Thus

$$P_{11} = \sigma^2 A_1 = .281 \times 10^{-4}$$

Since

$$A_1 = \beta g$$

$$g^2\sigma^2{}_{\beta} = \sigma A_1{}^2$$

Or

$$\sigma_{\beta} = 21 \times 10^3\,\sigma A_1 \text{ arc-second}$$

$$\sigma_{\beta} = 111 \text{ arc-seconds}$$

This appears to be the error in determining initial misalignment after measurements lasting 25 seconds.

Continuing let us evaluate the variance of the drift estimate. Here

$$P_{22} = \sigma^2{}_{A2} \doteq .461 \times 10^{-7}$$

Since

$$A_2 = g\frac{d}{2}$$

$$\sigma_d = \frac{2}{g}\,\sigma_{A2} = .044 \times 10^{-3} \text{ rad/sec.} = 9.05 \text{ deg/hr}$$

Here we note the extremely large uncertainty in drift. However, we know apriori that this uncertainty is very small. This knowledge can be used to reduce all variances in P_A. We consider an effective additional measurement.

$$y = H \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + \text{noise} \tag{3-152}$$

where

$$H = [\,0 \quad 1\,]$$

since only drift is measured. The variance of the noise in equation 3-152 is R and is very small compared to P_{22}. After this measurement is used P_A is reduced to

$$P_A = P_A - P_A H^T \; [HP_A H^T + R]^{-1} HP_A \tag{3-153}$$

Expanding we find

$$P_A = \begin{bmatrix} P_{11} - \dfrac{P_{12}^2}{P_{22}+R} & P_{12} - \dfrac{P_{12}P_{22}}{P_{22}+R} \\ P_{12} - \dfrac{P_{12}P_{22}}{P_{22}+R} & P_{22} - \dfrac{P_{22}^2}{P_{22}+R} \end{bmatrix} \tag{3-154}$$

Since the drift uncertainty is so small compared to P_{22}, the variance of A_1 now becomes essentially

$$\sigma A_1^{\,2} = .281 \times 10^{-4} - \frac{(.110 \times 10^{-5})^2}{.461 \times 10^{-7}} = .0146 \times 10^{-4}$$

and thus

$$\sigma_\beta = 25.6 \text{ arc-seconds}$$

This is a much more reasonable value. An alternate approach would note that an uncertainty of drift of even 1 meru = .015/arc sec/sec would cause an additional misalignment of less than an arc-second in the total measurement period. We may in effect continue our analysis using the maximum likelihood method to fit only a linear time term in order to make our error estimates.

Noise Spectrum

A typical velocity auto-correlation function for velocity is shown in equation 3-151 . We note however that the velocity measured is always the difference between an initial velocity and the one measured, i.e.

$$V_n = V_n - V_o$$

Thus, the autocorrelation is

$$\langle V_m \; V_n \rangle = \langle V_m, V_n \rangle - \langle V_m, V_o \rangle - \langle V_n, V_o \rangle + \langle V_o, V_o \rangle \qquad (3\text{-}155)$$

Results

Sampling Time

For a velocity noise auto-correlation given by equation 3-151 , a series of computer runs were made investigating the effect of sampling time. The noise center point period was 3.14 seconds. Results shown in Figure 28 indicate that for sampling times below 1 second, results are very nearly the same. In Figure 29 we see that large errors exist when the sampling period exceeds the center point period. One second sampling time will be used for all the following results.

Noise Model

The velocity noise auto-correlation will be represented by

$$R(n) = R_o \, \delta(o) + R_1 \; e^{-n\tau/T} \; \cos Wn\tau \qquad (3\text{-}156)$$

The study looks at the miserection error variance after 60 seconds of measurements for a variety of noise parameters. The filtering model here matches the true noise model. The model allows for a white noise component, R_o, and a sinusoidal term. In general, R_1 will always be .5 (meters/sec)^2.

a) Effect of Frequency

R_o	W	T	One Sigma Miserection Error
0	1.5	200	8.77 arc-seconds
0	2.0	200	7.03
0	2.5	200	6.19

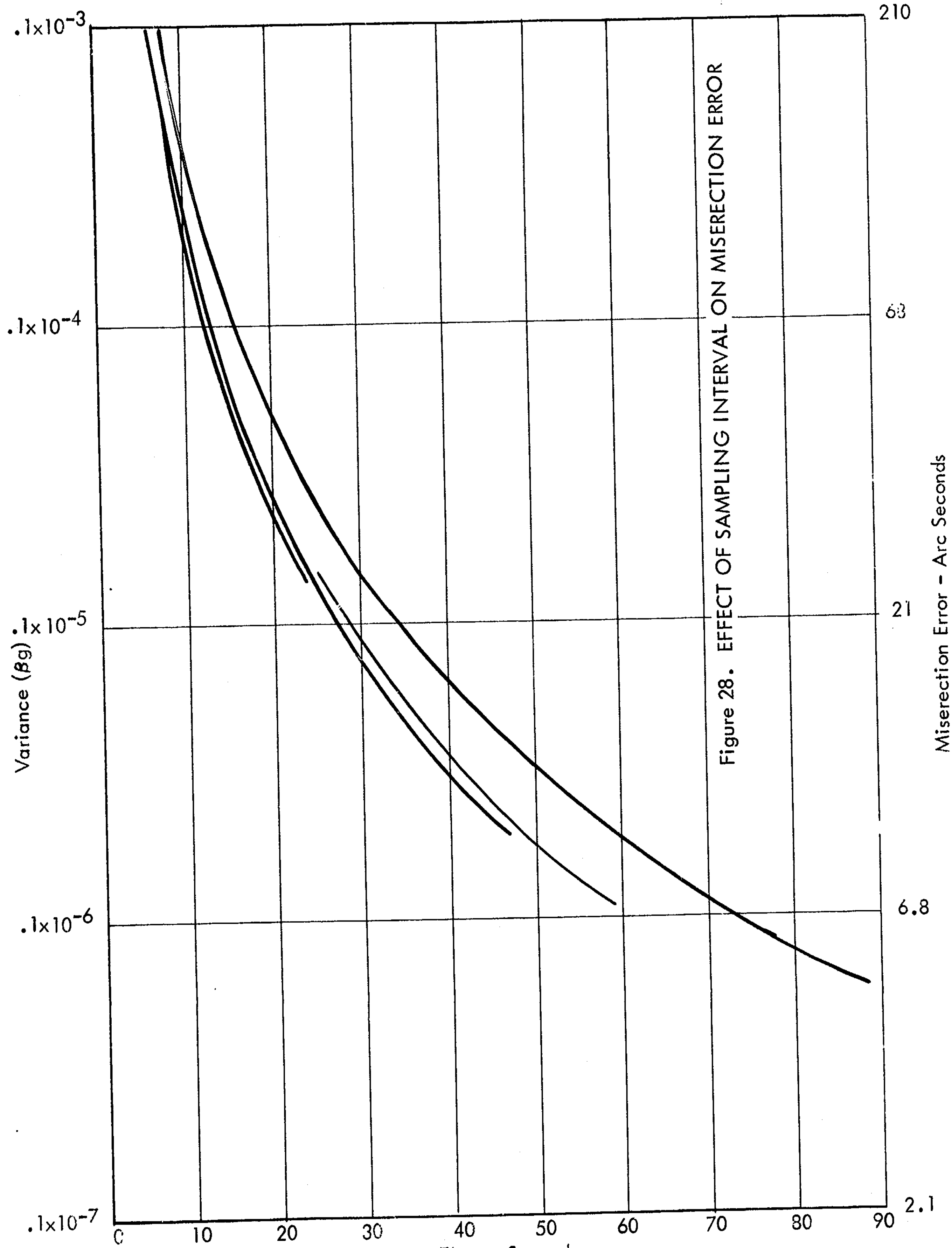


Figure 28. EFFECT OF SAMPLING INTERVAL ON MISERECTION ERROR

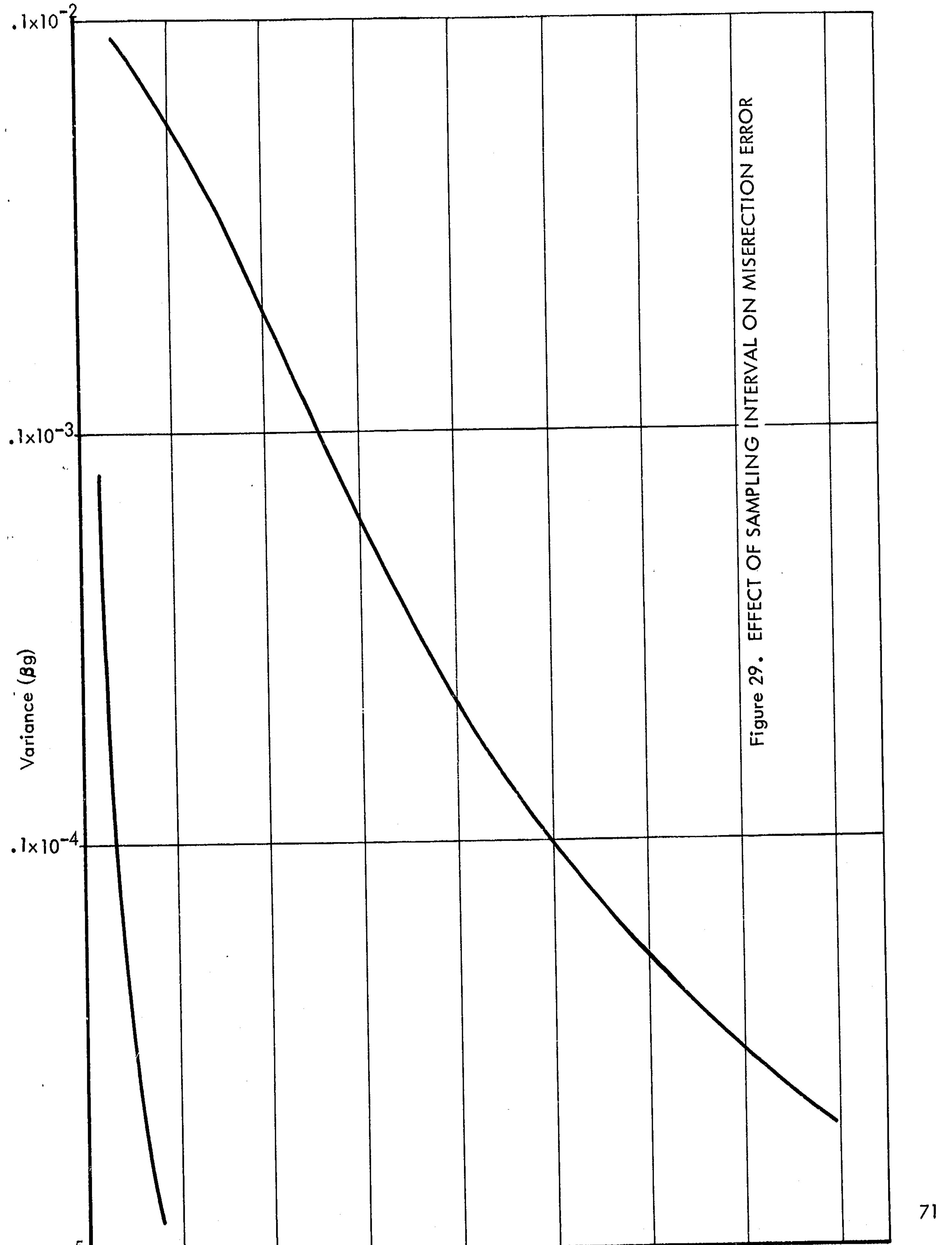


Figure 29. EFFECT OF SAMPLING INTERVAL ON MISERECTION ERROR

b) Effect of Correlation Time

R_o	W	T	One Sigma Miserection Error
0	2	200	7.03 arc-seconds
0	2	100	10.30
0	2	20	22.2
0	2	2	67.5
0	2	0	114

c) Effect of White Noise Component

R_o	W	T	One Sigma Miserection Error
0	2	200	7.03
.1	2	200	51.8
.2	2	200	72.5

d) All White Noise

$R_o = .5 \quad R_1 = 0$ 114

Here we see the best results which can be obtained after 60 seconds of erection time. For high accuracy we require that the noise be highly correlated.

Filtering Effects

We now consider true noise to be represented by equation 3-156 with

$R_o = 0$

$R_1 = .5\ (\text{meter/sec})^2$

$W = 2$ radians/sec.

$T = 200$ seconds

and study the effect of a filtering model which does not match the true noise. Parameter shown below are for the filter model

R_o	R_1	W	T	One Sigma Miserection Error-Arc Sec.	Comments
0	.5	2	200	7.03	Perfect Model
0	.5	2.5	200	9.7	Mismatched Frequency
0	.5	1.5	200	13.0	
0	.5	2	100	7.03	Mismatched Correlation Time
.1	.5	2	200	7.80	White noise Component
.2	.5	2	200	8.18	
.5	.5	2	200	8.85	
.5	0			17.5	Least squares filter

We note that it is desirable to have the filter match the noise as much as possible. When the characteristics of the true noise are uncertain, one can guess safely by using a higher frequency and shorter correlation time for the filtering model. A small amount of white noise should also be used to model the accelerometer quantization.

The data below investigates the effect of two or more filter parameters which do not match the true noise. These lead to the same conclusions as the previous data.

R_o	R_1	W	T	One Sigma Miserection Error-Arc. Sec.	Comments
.1	.5	2.5	200	16.3	White noise component and mismatched frequency
.2	.5	2.5	200	15.3	
.1	.5	1.5	200	13.5	
.2	.5	1.5	200	13.1	
.1	.5	2	20	7.5	White noise component and mismatched correlation time
.2	.5	2	20		
0	.5	2.5	20	9.2	White noise component, mismatched frequency and correlation time
.1	.5	2.5	20	10.7	
.2	.5	2.5	20	12.8	

Figure 30 compares the error variance when using a least square filter with a perfect model.

Weighting Function

Figure 31 shows the coefficients that the velocity data points are multiplied by for two filter models. These are presented primarily to show the affect of acceleration circuitry in producing an early or late pulse. The error caused is the difference between weighting coefficients for two adjacent points. This is about $.6 \times 10^{-4}$. For a velocity quantization of .02 meter/sec., this causes an error of

$$\beta = \frac{.02 \times .6 \times 10^{-4}}{9.8} = .122 \times 10^{-6} \text{ radians} = .025 \text{ seconds which is negligible.}$$

Relation to Kalman Filter

When the dynamic model, equation 3-136 and 3-137 is proper, the Kalman filter will give identical results as the maximum likelihood filter.

If apriori data is available as discussed in the example here, proper use of this data conserves similarity between these two filtering methods. The Kalman filter does allow a more realistic dynamic model in which earth's rotation is considered without adding complexity to the computations and more easily allows additional data such as the azimuth measurement to be entered into the filtering process.

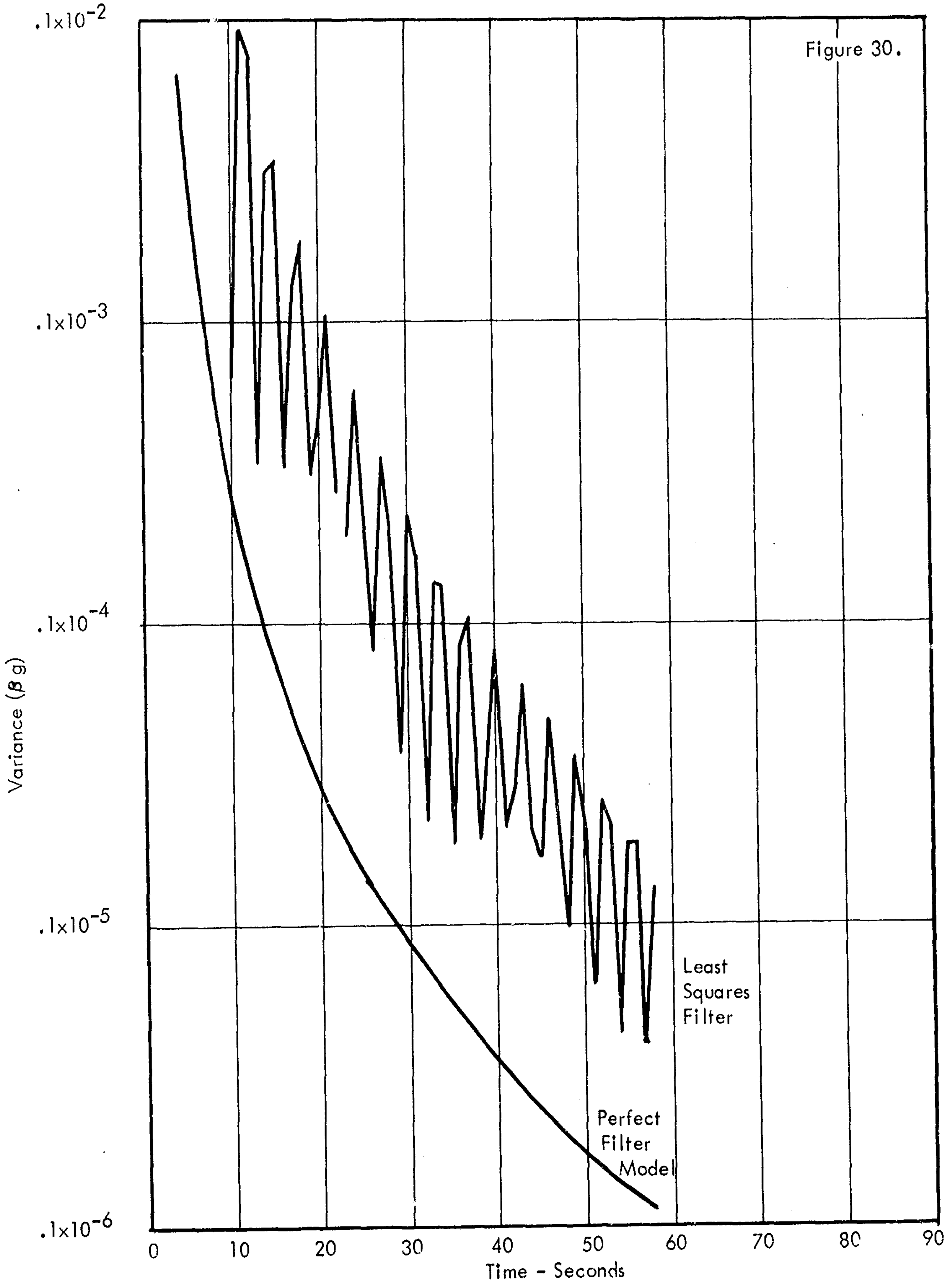


Figure 30.
.1x10^{-2}
.1x10^{-3}
.1x10^{-4}
.1x10^{-5}
.1x10^{-6}
Variance (β g)
0
10
20
30
40
50
60
70
80
90
Time - Seconds
Least Squares Filter
Perfect Filter Model

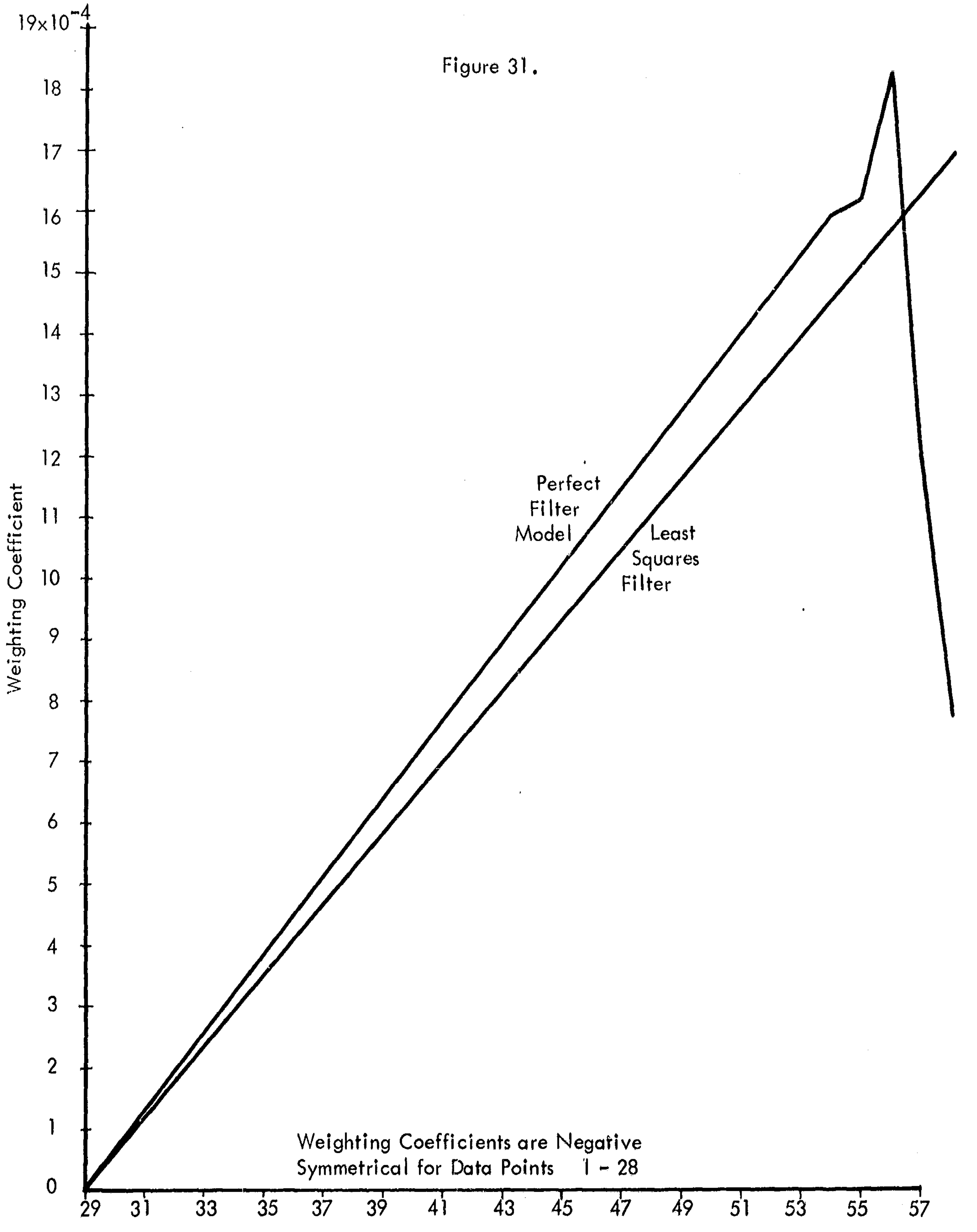


Figure 31.
19x10^-4
18
17
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
Weighting Coefficient
Perfect Filter Model
Least Squares Filter
Weighting Coefficients are Negative
Symmetrical for Data Points 1 - 28
29
31
33
35
37
39
41
43
45
47
49
51
53
55
57

F. CORRECTION FOR BIAS ERRORS IN THE LEAST SQUARES FILTER

It was pointed out in Section III-C that a bias exists in the angle estimates for both vertical and azimuth. The azimuth bias is due to vertical error bias which in turn is due to earth's rate crosscouplings. An expression for an optimal correction matrix was derived and it was pointed out that the simpler matrix derived for a least squares filter would provide adequate accuracy. The purpose of this section is to derive the simple correction matrix.

This matrix is based on the assumption that the difference between the actual and measured angles can be expressed in terms of the actual angles, the geometry of the theodolite measurement, and earth's rate. The coordinates systems used are shown in Figure 18.

The azimuth measurement is given by

$$m_1 = \alpha + \tan \gamma_o \left[\beta \sin \epsilon + \gamma \cos \epsilon\right] \qquad 3\text{-}157$$

Where

- α is the actual azimuth angle
- β is the actual miserection angle about the j axis
- γ is the actual miserection angle about the Z axis
- γ_o is the theodolite elevation angle
- ϵ is the angle between the theodolite line of sight and the Z_o axis

Using the results of Appendix C (Equations C-4 and C-5) the vertical measurement can be written as

$$m_2 = 1/2\, \omega_Z \alpha T + \beta - 1/2\, \omega_x \gamma T \qquad 3\text{-}158.$$

$$m_3 = -1/2\, \omega_y \alpha T + 1/2\, \omega_x \beta T + \gamma \qquad 3\text{-}159$$

Where T equals the time of the measurement ω_x, ω_y, ω_z are the components of earth's rate about the x_o, y_o, and z_o axes.

Eauations 3-157, 3-158 and 3-159 can be solved simultaneously to yield

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} \qquad 3\text{-}160$$

The matrix coefficients are given by

$$a_{11} = (1 + 1/4\, \omega_x^2\, T^2)/\Delta$$

$$a_{12} = (1/2\, C_2 \omega_x T - C_1)/\Delta$$

$$a_{13} = (-C_2 - 1/2\, C_1\, \omega_x\, T)/\Delta$$

$$a_{21} = (1/4\, \omega_x\, \omega_y\, T^2 - 1/2\, \omega_Z\, T)/\Delta$$

$$a_{22} = (1 + 1/2\, C_2 \omega_y T)/\Delta$$

$$a_{23} = (1/2\, \omega_x T + 1/2\, C_2 \omega_z T)/\Delta$$

$$a_{31} = (1/4\, \omega_x \omega_z\, T^2 + 1/2\, \omega_y T)/\Delta$$

$$a_{32} = (-1/2\, C_1\, \omega_y T - 1/2\, \omega_x T)/\Delta$$

$$a_{33} = (1 - 1/2\, C_1\, \omega_Z T)/\Delta$$

$$\Delta = 1 + 1/2T\, [\, C_2\, \omega_y - C_1\, \omega_Z\,] + 1/4\, T^2\, \omega_x\, [\, C_1\, \omega_y + C_2 \omega_Z + \omega_x\,]$$

$$C_1 = \tan \gamma_o \sin \epsilon$$

$$C_2 = \tan \gamma_o \cos \epsilon$$

Equation 3-160 is completely valid only if there are no errors in the vertical angle measurements except for earth's rate coupling effects. There will be errors due to imperfect filtering of sway velocity.

However, examination of the magnitude of the coefficients of the matrix indicates that these errors should have only a small effect on the correction applied. To confirm this conclusion, two runs were made with the Erection and Alignment Simulation Program.

In each of these runs, an initial condition of 2 degrees on each of the three axes was assumed. In one case, a zero sway velocity noise was used, and in the other case a sway velocity noise of 0.5 m/sec rms at a center frequency of 2 radians/second with a correlation time of 200 seconds was used. The results are tabulated in Tables 1 and 2.

Table 1
Angular Errors Before and After Correction for
Earth's Rate Coupling (Zero Noise)

	Angular Error (secs of arc)					
	Kalman Filter Estimates			Least Squares Estimates		
	α	β	γ	α	β	γ
Original Estimate	--	-7.6	-8.3	--	-7.1	-7.9
Estimate after Correction	0.3	-.7	-1.1	.1	-.2	-.7
Amount of Correction Applied	--	6.9	7.2	--	6.9	7.2

Table 2
Angular Errors Before and After Correction for
Earth's Rate Coupling (0.5 m/sec noise)

	Angular Error (secs of arc)					
	Kalman Filter Estimates			Least Squares Estimates		
	α	β	γ	α	β	γ
Original Estimate	--	10.7	-.5	--	14.5	6.8
Estimate after Correction	-8.6	17.6	6.7	-10.5	21.4	13.9
Amount of Correction Applied	--	6.9	7.2	--	6.9	7.1

It can be seen that with zero noise the correction scheme removes the error due to earth's rate coupling to an accuracy of about 1 second of arc. Secondly, it can be seen that the correction applied to remove the earth's rate coupling varies only by about 0.1 seconds of arc when the noise has a value of 0.5 m/sec. Thus, it can be concluded that the correction matrix will accurately remove the errors due to earth's rate coupling even in the presence of noise. Unfortunately, in the run shown, the vertical errors both increased when the correction was made. In this case, the errors due to sway velocity noise and the errors due to earth's rate coupling tended to cancel. They will not tend to cancel each other in all cases, however, and in a statistical sense the answer has to be improved by removing the bias due to earth's rate coupling. The concept of the use of the correction matrix was developed too late in this study to allow a rigorous statistical analysis of the improvement in performance.

IV. DESCRIPTION OF THE ERECTION AND ALIGNMENT SIMULATION PROGRAM

A. THE PROGRAM EQUATIONS

The erection and alignment program solves the Kalman filter equations given by equations 3-94 through 3-100 and the least squares filter equations given by equationC-3 for each channel. In addition, the program simulates a theodolite measurement at the final time. This is done by extrapolating the initial angles to time t_f by the equation.

$$\theta(t_f) = \phi_{11}(t_f, t_o)\,\theta(t_o)$$

Where $\phi_{11}(t_f, t_o)$ is approximated by:

$$\phi_{11}(t_f, t_o) = \left[I - \Omega(t_f - t_o) \right]$$

The theodolite measurement is then given by:

$$m_1 = \alpha(t_f) + C_1\,\beta(t_f) + C_2\gamma(t_f)$$

Where $C_1 = \tan\gamma_o \sin\epsilon$

$$C_2 = \tan\gamma_o \cos\epsilon$$

The program calculates the correction matrix derived in Section III-F and then calculates the final estimates of α, β, and γ.

B. DEFINITION OF THE PROGRAM CONSTANTS

There are 16 floating point constants which must be read in. They are called C(1) through C(16). They are defined by

Program Constant		Definition
C(1)	σ_θ	- rms angle error (arc seconds)
C(2)	σ_v	- rms sway velocity (m/sec)
C(3)	f_o	- sway velocity center frequency (cps)
C(4)	τ	- sway velocity correlation time (seconds)
C(5)	g	- gravity $(m/sec)^2$
C(6)	Δ	- acceleration quantization level (m/sec)

Program Constant			Definition
C(7)	Δt	-	time between measurements (seconds)
C(8)	t_f	-	final time (seconds)
C(9)	ω	-	earth's rotation rate (rad/sec)
C(10)	L	-	latitude of launch site (degrees)
C(11)	B	-	target bearing (west of north) (degrees)
C(12)	γ_o	-	theodolite elevation angle (degrees)
C(13)	ϵ	-	angle between Z_1 axis and theodolite LOS (degrees)
C(14)	$\alpha(t_o)$		
C(15)	$\beta(t_o)$	-	initial misalignment angles in degrees
C(16)	$\gamma(t_o)$		

α, β, and γ are defined by Figure 18. When B is equal to zero, the Z axis is north.

C. INPUT CARD FORMAT

The format of the input cards is listed in Table 4.

Table 4
INPUT CARD FORMAT

Card No.	Variable	Format	Description
1	NNR UN	I2	NUMBER OF RUNS
2	IPRNT	I2	=1 - PRINT FILTER GAINS AND CORRECTION MATRIX, = 0 - BYPASS
3	C(1)	E20.8	C'S DESCRIBED ABOVE
4	C(2)		
5	C(3)		
6	C(4)		
7	C(5)		
8	C(6)		
9	C(7)		
10	C(8)		
11	C(9)		
12	C(10)		
13	C(11)		
14	C(12)		
15	C(13)		
16	C(14)		
17	C(15)		
18	C(16)	E20.8	

Table I (Continued)

Card No.	Variable	Format	Description
19		80H	HOLLERITH IDENTIFICATION
20	NUMBER, DA,TE	I2, A6, A2	CASE NUMBER, DATE
21	T(IS), XD(1), XD(2)	3X, E14.7, 22X, E14.7, 4X, E14.7	TEST DATA - Time, y velocity, Z velocity
↓	↓		↓

MULTIPLE CASES BEGIN WITH CARD 19, IDENTIFICATION

D. PROGRAM OUTPUT

The program output consists of a printout. First the program constants defined in Part B of this section are printed out in three rows. The first row contains C(1) through C(6). The second row contains C(7) through C(12) and the third row contains C(13) through C(16). Then the estimates of β and γ given by the Kalman and least squares filter equations are printed out at each sampling time. Then the final angle estimates at the end of the filtering period are printed out. These estimates are calculated using the correction matrix of Section III-F. The actual angles at the end of the filtering period are then printed out. If the printout of the filter gains option is chosen, the 4 gains used in the Kalman filter at each sampling time and the elements of the correction matrix used to determine the final angle estimates at the end of the filtering period are printed out.

V. DEFINITION OF COMPUTER REQUIREMENTS

A. ASSUMPTIONS

1) These requirements for storage and execution time are based on the characteristics of the RCA 110-A computer. The execution time estimates assume fixed point calculations.
2) It is assumed that certain calculations can be made each time a data point is received. This is not a necessary assumption but it reduces the required storage since each data point does not have to be stored.
3) In determining the azimuth requirements, the multiplication ${}^{1}C_{2}\,{}^{2}C_{4}$ is assumed to be performed using a conventional matrix multiplication algorithm with several loops. Since the other matrices contain many ones and zeros, the multiplications involving them do not use the loop type algorithm in order to save execution time. A slight penalty of about 50 storage words is incurred by this approach.
4) When defining both the storage required and the execution time, it is assumed that an increase of 50% in the arithmetic instructions will be required for scaling.
5) No allocation has been included for program self-checking provisions.
6) The filter weights are assumed to be precalculated and stored.
7) The program used for determing the requirements is configured so that all read-in constants are saved and initial conditions of variables and index registers are set without further read-in if more than one pass is required due to a hold on the launch sequence.
8) The CTMC velocity registers are assumed to be set to zero before start.

B. EQUATIONS

1) Azimuth - The equations used are based on those shown in AIAA paper No. 67-556 "Initial Alignment of a Strapdown Inertial Reference and Navigation System" by Hans F. Kennel.

$$ {}^{1}C_{6} = {}^{1}C_{2}\ {}^{2}C_{4}\ {}^{4}C_{5}\ {}^{5}C_{6} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} $$

The elements of ${}^1C_2$ are obtained from the CTMC. The elements of ${}^2C_4$ and ${}^5C_6$ are fixed constants which are read in. The elements of ${}^4C_5$ are calculated from the porro prism encoder angle and a fixed misalignment.

After calculating the elements of ${}^1C_6$ the following calculations are required.

$$\sin \gamma_p = \frac{a_{11} \sin \gamma_0 + a_{12} \sqrt{a_{12}^2 - \sin^2 \gamma_0 + a_{11}^2}}{a_{11}^2 + a_{12}^2}$$

$$\cos \gamma_p = \sqrt{1 - \sin^2 \gamma_p}$$

$$L_{y1} = -a_{21} \sin \gamma_p + a_{22} \cos \gamma_p$$

$$L_{Z1} = -a_{31} \sin \gamma_p + a_{32} \cos \gamma_p$$

$$m_1 = \frac{\pi}{2} + \tan^{-1} \frac{L_{y1}}{L_{Z1}}$$

m_1 is the azimuth measurement to be used in the correction equations.

2) Kalman Filter Equations for Level - The equations to be solved are equations 3-94 to 3-97 in Section III. The K* (t_i) are precalculated stored constants and are calculated by the Erection and Alignment Simulation program for each data point. The equations are of a recursive form and the four components of each $\hat{x}$ are calculated each time data is received.

3) Least Squares Equations for Level - The equation to be solved for each channel is given by equation C-3. Each time a data point is received the ΣV_i and $\Sigma t V_i$ are calculated. At the end of the filtering period the two summations are multiplied by stored constants and the resulting products added to provide an angle estimate.

4) Corrections Equations for Bias Errors - These equations are given by equation 3-160 of Section III-F. The a's are precalculated and stored constants. m_1 is obtained from the azimuth calculation. m_1 and m_2 are obtained from the level equations.

C. COMPUTER REQUIREMENTS

The storage requirements are tabulated in Table 5.

Table 5

STORAGE LOCATIONS REQUIRED

	Program Storage	Stored Constants	Scratch Pad	Scaling Instructions	Total
Azimuth	171	27	43	20	261
Kalman Filter	77	247*	16	13	353
Least Squares Filter	34	5	8	4	51
Correction Equations	19	9	8	2	38

* Kalman filter assumed 60 data points. For each additional data point, four additional storage locations are required to store weights.

Total storage requirements, assuming scratch pad is shared, are 636 locations if the Kalman Filter is used and 342 locations if the Least Squares Filter is used.

The execution times are listed below.

Azimuth

0.085 seconds plus 1 pass through the sin/cosine subroutine, 2 passes through the square root subroutine and 1 pass through the arc tangent subroutine is required. The time to execute these subroutines in RCA-110A computer was not listed on the books provided.

Kalman Filter

When each data point is received, 0.02 seconds plus 1 pass through the sin/cosine subroutine is required. In addition 0.00086 seconds is required to zero initial conditions at start of the filtering process.

Least Squares Filter

When each data point is received 0.0048 seconds is required. In addition, 0.003 seconds is required to zero initial conditions at the start of the filtering process and to perform calculations at the end of the filtering process.

Correction Equations

0.0095 seconds is required.

APPENDIX A -

FILTERING PROBLEM WHEN ERECTING WITH STRAPED DOWN ACCELEROMETERS

PURPOSE

The purpose of this appendix is first to derive the equations describing the output of strapped down accelerometers in a rotating and translating vehicle. The magnitude of the various terms in these equations are then discussed and the terms which must be filtered out are determined.

DERIVATION OF THE ACCELEROMETER OUTPUTS

The coordinate systems to be used are shown in Figure A-1. The $\bar{i}_o, \bar{j}_o, \bar{k}_o$ system is an inertial system with origin at the center of the earth. The $\bar{i}, \bar{j}, \bar{k}$ system is earth fixed with its origin at the center of the earth. The $\bar{i}', \bar{j}', \bar{k}'$ is missile fixed with its origin at the center of rotation of the missile. The position vector of the accelerometers is given by $\bar{R}_o$ in the inertial system and is made up of the components $\bar{R}$, the vector from the center of the earth to the center of rotation of the missile and $\bar{r}$, the vector from the missile center of rotation to the accelerometers.

$$\text{Then } \bar{R}_o = \bar{R} + \bar{r}$$

$$= R_x \bar{i} + R_y \bar{j} + R_z \bar{k} + r_x \bar{i}' + r_y \bar{j}' + r_z \bar{k}' \qquad \text{(A-1)}$$

The acceleration of the accelerometers is given by the second derivative of $\bar{R}_o$ with respect to inertial space.

$$\dot{\bar{R}}_o = \dot{R}_x \bar{i} + \dot{R}_y \bar{j} + \dot{R}_z \bar{K} + R_x \dot{\bar{i}} + R_y \dot{\bar{j}} + R_z \dot{\bar{k}} + \dot{r}_x \bar{i}' + \dot{r}_y \bar{j}'$$
$$\dot{r}_z \bar{k}' + r_x \dot{\bar{i}}' + r_y \dot{\bar{j}}' + r_z \dot{\bar{k}}' \qquad \text{(A-2)}$$

The first group of three terms in A-2 is the velocity of the origin of the primed system with respect to the earth and will be called $\bar{V}$. The second group of three terms in A-2 is the velocity due to the rotation of the earth and can be written $\bar{W}_E \times \bar{R}$. The third group of three terms in A-2 is the velocity with respect to the missile-fixed system and, thus, is zero. The fourth group of three terms in the velocity with respect to the origin of the primed system due to rotation of the primed system and can be written $(\bar{W}_E + \bar{W}_m) \times \bar{r}$. Here $\bar{W}_E$ is the rotation of the earth and $\bar{W}_m$ is the rate of rotation of the missile with respect to the earth.

Hence

$$\dot{\bar{R}}_o = \bar{V} + \bar{W}_E \times (\bar{R} + \bar{r}) + \bar{W}_m \times \bar{r} \tag{A-3}$$

Then

$$\ddot{\bar{R}}_o = \dot{\bar{V}} + \bar{W}_E \times (\dot{\bar{R}} + \dot{\bar{r}}) + \dot{\bar{W}}_m \times \bar{r} + \bar{W}_m \times \dot{\bar{r}} \tag{A-4}$$

but

$$\dot{\bar{V}} = \ddot{R}_x \bar{i} + \ddot{R}_y \bar{j} + \ddot{R}_z \bar{K} + \dot{R}_x \dot{\bar{i}} + \dot{R}_y \dot{\bar{j}} + \dot{R}_z \dot{\bar{K}} = \bar{A} + (W_E \times \bar{V})$$

Where $\bar{A}$ is the acceleration of the origin of the primed system with respect to the earth, then A-4 can be written

$$\begin{aligned}\ddot{\bar{R}}_o &= \bar{A} + (\bar{W}_E \times \bar{V}) + \bar{W}_E \times \left[\bar{V} + (\bar{W}_E \times \bar{R})\right] + \bar{W}_E \times \left[(\bar{W}_E + \bar{W}_m) \times \bar{r}\right] \\ &+ \dot{\bar{W}}_m \times \bar{r} + \bar{W}_m \times \left[(\bar{W}_E + \bar{W}_m) \times \bar{r}\right] = \bar{A} + 2\,(\bar{W}_E \times \bar{V}) + \bar{W}_E \times \\ &\left[\bar{W}_E \times (\bar{R} + \bar{r})\right] + \bar{W}_m \times (\bar{W}_m \times \bar{r}) + \bar{W}_E \times (\bar{W}_m \times \bar{r}) + \bar{W}_m \times \\ &(\bar{W}_E \times \bar{r}) + \dot{\bar{W}}_m \times \bar{r}\end{aligned} \tag{A-5}$$

If $\bar{W}_m$ is written as

$$\bar{W}_m = W_{mx}\bar{i}' + W_{my}\bar{j}' + W_{mz}\bar{K}'$$

Then

$$\begin{aligned}\dot{\bar{W}}_m &= \dot{W}_{mx}\bar{i}' + \dot{W}_{my}\bar{j}' + \dot{W}_{mz}\bar{K}' + W_{mx}\dot{\bar{i}}' + W_{my}\dot{\bar{j}}' + W_{mz}\dot{\bar{K}}' \\ &= \overset{\tau}{\bar{W}}_m + (\bar{W}_m + \bar{W}_E) \times \bar{W}_m\end{aligned}$$

$$\dot{\bar{W}}_m = \overset{\tau}{\bar{W}}_m + \bar{W}_E \times \bar{W}_m \tag{A-6}$$

Where $\overset{\tau}{\bar{W}}_m$ is the angular acceleration of the primed system with respect to the earth. Using A-6, the last three terms in equation A-5 become

$$\begin{aligned}\bar{X} &= \bar{W}_E \times (\bar{W}_m \times \bar{r}) + \bar{W}_m \times (\bar{W}_E \times \bar{r}) + \dot{\bar{W}}_m \times \bar{r} = \overset{\tau}{W}_m \times \bar{r} + (\bar{W}_E \times \bar{W}_m) \times \bar{r} \\ &+ \bar{W}_E \times (\bar{W}_m \times \bar{r}) + \bar{W}_m \times (\bar{W}_E \times \bar{r})\end{aligned} \tag{A-7}$$

Then using the relationships

$$(\bar{P} \times \bar{Q}) \times \bar{R} = (\bar{R} \cdot \bar{P})\,\bar{Q} - (\bar{R} \cdot \bar{Q})\,\bar{P}$$

$$\bar{R} \times (\bar{P} \times \bar{Q}) = (\bar{R} \cdot \bar{Q})\,\bar{P} - (\bar{R} \cdot \bar{P})\,\bar{Q}$$

$$\bar{X} = \dot{\bar{W}}_m \times \bar{r} + (\bar{W}_E \cdot \bar{r})\,\bar{W}_m - (\bar{r} \cdot \bar{W}_m)\,\bar{W}_E + (\bar{W}_E \cdot \bar{r})\,\bar{W}_m - (\bar{W}_E \cdot \bar{W}_m)\,\bar{r}$$

$$+ (\bar{W}_m \cdot \bar{r})\,\bar{W}_E - (\bar{W}_m \cdot \bar{W}_E)\,\bar{r} = \dot{\bar{W}}_m \times \bar{r} + 2$$

$$\times \left[(\bar{W}_E \cdot \bar{r})\,\bar{W}_m - (\bar{W}_E \cdot \bar{W}_m)\,\bar{r}\right]$$

$$\bar{X} = \dot{\bar{W}}_m \times \bar{r} + 2\; W_E \times (\bar{W}_m \times \bar{r}) \qquad \text{(A-8)}$$

Using A-7 and A-8 in A-5 and the fact that the accelerometers read $\bar{g} + \ddot{\bar{R}}_o$ yields

$$\begin{matrix}\text{Accelerometer}\\ \text{Output}\end{matrix} = g + \left(W_E \times \left[\bar{W}_E \times (\bar{R} \times \bar{r})\right]\right) + \bar{A} + \dot{\bar{W}}_m \times \bar{r} + \bar{W}_m \times (\bar{W}_m \times \bar{r})$$

$$+ 2\,\bar{W}_E \times \left[(\bar{W}_m \times \bar{r}) + \bar{V}\right] \qquad \text{(A-9)}$$

DISCUSSION OF THE ACCELEROMETER OUTPUT

Of interest here are the horizontal components of equation A-9 since these are the outputs of the transformation computer that will be used to erect the computing coordinate system in the computer. When the system is erected, these outputs should be zero. Therefore, it is any non-zero horizontal components of equation A-9 which must be filtered out.

The first two terms of equation A-5 are due to gravitational attraction and the centrifugal acceleration of the earth and are essentially directed along the local vertical ($R >> r$). The second three terms represent the acceleration relative to the earth and are caused by missile sway in the case of interest here. The last term in equation A-9 is the Coriolis acceleration and is proportional to the velocity relative to the earth.

For missile sway then, both the acceleration and velocity relative to the earth will be essentially horizontal and have magnitudes related by the equation $A = WV$

Where:
- A = magnitude of the acceleration
- V = magnitude of the velocity
- W = frequency of sway motion

Then the ratio of the horizontal acceleration to the Coriolis acceleration is equal to or greater than $\frac{W}{2 W_e}$. For a W of 1-radian-per-second, this ratio is on the order of 10^4.

Therefore, the Coriolis acceleration can be neglected and the quantity to be filtered is the output of the accelerometer due to sway acceleration. Writing the sway acceleration as

$$a = a \sin (Wt + \emptyset) \qquad \text{(A-10)}$$

The integrated output of the accelerometer; i.e. the velocity indicated by the accelerometer is $V = -\frac{a}{W} \cos (Wt + \emptyset) + \frac{a}{W} \cos \emptyset$ (A-11)

It should be noted that even though the input acceleration is oscillatory, the output of the accelerometer has a constant term in its integral. The magnitude of this term is a function of the phase angle of the sway acceleration at the time the integration procedure is started. The filtering technique must then be concerned not only with the elimination of oscillatory error terms but also with the elimination of constant error terms.

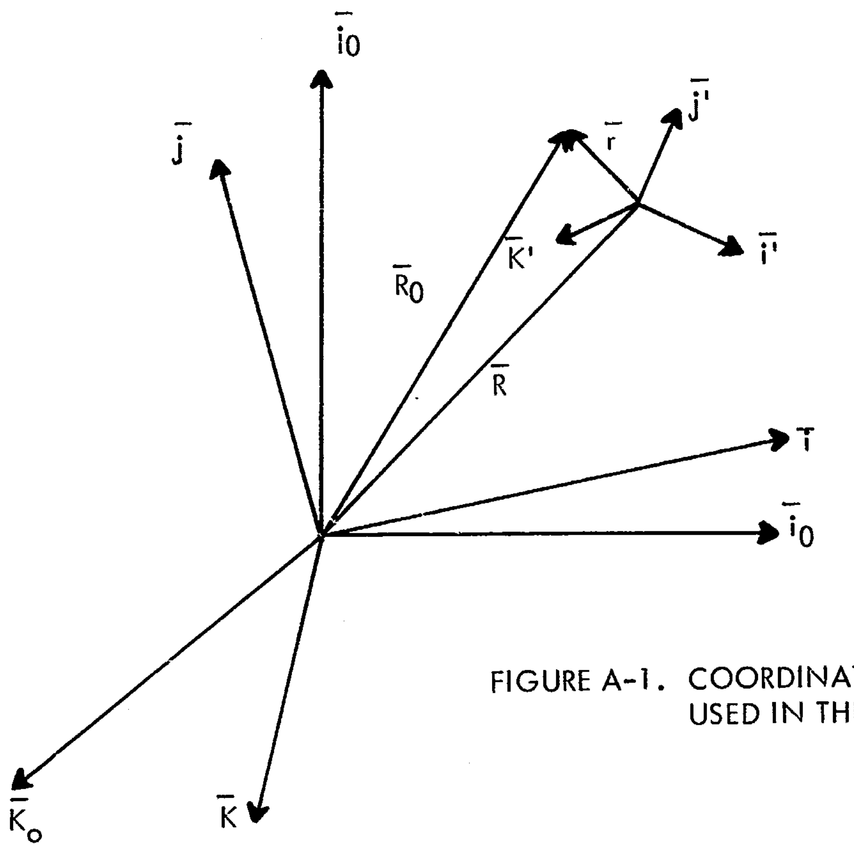


FIGURE A-1. COORDINATE SYSTEMS USED IN THE ANALYSIS

APPENDIX B
OPTICAL ALIGNMENT EQUATIONS

The purpose of this appendix is to explain the azimuth angle equations contained in the report "Initial Alignment of a Strapped-down Inertial Reference and Navigation System" by Hans F. Kennel. Specifically, equation 16 and the equation at the top of page seven for δA, the azimuth error caused by leveling error will be derived. This work was done to assure that we had a proper understanding of the coordinate systems involved and the relationship between azimuth and leveling errors.

Figure B-1 shows the relationships between the various coordinate systems used in Kennel's report under the conditions that the system has been leveled (that is: coordinate systems 1 and 3 are coincident). The azimuth angle measurement desired is α_1, the angle between the theodolite reference and the Y_1 axis. The components of the line of sight in the 6 system are given by

$$
\begin{aligned}
L_{x6} &= -\sin \gamma_p \\
L_{y6} &= \cos \gamma_p \\
L_{z6} &= 0
\end{aligned}
$$

Then the components in the 1 system are given by

$$
\begin{bmatrix} L_{x1} \\ L_{y1} \\ L_{z1} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} -\sin \gamma_p \\ \cos \gamma_p \\ 0 \end{bmatrix} \qquad \text{(B-1)}
$$

where the a matrix is the transformation matrix between the 1 and 6 systems. When the 1 system is level, this matrix is a function of missile tilt, the porro-prism encoder angle, and the known misalignment δ's.

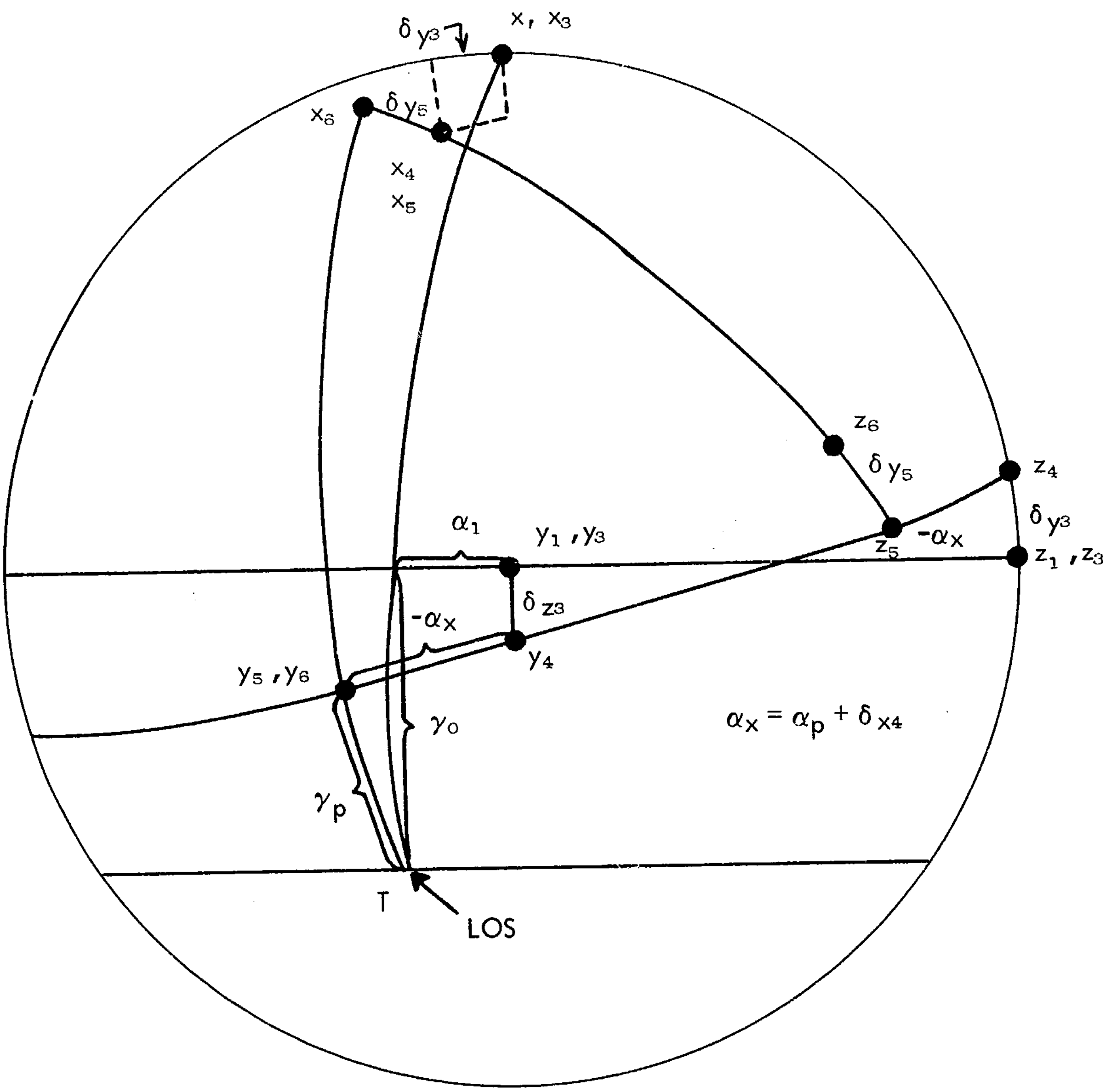
But L_{y1}, and L_{z1} can also be written in terms of the angles α_1 and γ_0, the elevation of the line of sight in the leveled coordinate system

$$L_{y1} = \cos \gamma_0 \cos \alpha_1$$

$$L_{z1} = -\cos \gamma_0 \sin \alpha_1$$

then

$$\frac{L_{y1}}{L_{z1}} = -\text{ctn}\, \alpha_1$$



δ_{y3}
x_1, x_3
x_6
δ_{y5}
x_4
x_5
z_6
δ_{y5}
z_4
δ_{y3}
α_1
y_1, y_3
z_5
$-\alpha_x$
z_1, z_3
δ_{z3}
$-\alpha_x$
y_4
y_5, y_6
γ_o
$\alpha_x = \alpha_p + \delta_{x4}$
γ_p
T
LOS

FIGURE B-1.

but

$$\tan(\alpha_1 - \pi/2) = \frac{\tan \alpha_1 \ -\tan \pi/2}{1 + \tan \pi/2 \ \tan \alpha_1} = -\mathrm{ctn}\ \alpha_1$$

$$\therefore \quad \alpha_1 = \pi/2 + \tan^{-1} \frac{L_{y1}}{L_{z1}} \tag{B-2}$$

which is equation 16 of Kennel's report.

The remaining thing to be determined is the error in the measurement when system 1 is not level. In Figure B-2 the non-level system 1 is denoted by primes and is mis-leveled by small angles α about z_1 and β about Y_1.

For small angles, the coordinate transformation matrix from the 1 system to the 1' system is:

$$C_1^{1'} = \begin{bmatrix} 1 & \gamma & -\beta \\ -\gamma & 1 & 0 \\ \beta & 0 & 1 \end{bmatrix}$$

but

$$\begin{bmatrix} L_{x1} \\ L_{y1} \\ L_{z1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_1 & +\sin\alpha_1 \\ 0 & -\sin\alpha_1 & \cos\alpha_1 \end{bmatrix} \begin{bmatrix} -\sin\gamma_0 \\ \cos\gamma_0 \\ 0 \end{bmatrix} \tag{B-3}$$

so

$$\begin{bmatrix} L_{x1}' \\ L_{y1}' \\ L_{z1}' \end{bmatrix} = \begin{bmatrix} 1 & \gamma & -\beta \\ -\gamma & 1 & 0 \\ \beta & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_1 & +\sin\alpha_1 \\ 0 & -\sin\alpha_1 & \cos\alpha_1 \end{bmatrix} \begin{bmatrix} -\sin\gamma_0 \\ \cos\gamma_0 \\ 0 \end{bmatrix} \tag{B-4}$$

comparison of B-3 and B-4 indicates that

$$L_{y1}' = L_{y1} + \gamma \sin\gamma_0$$
$$L_{z1}' = L_{z1} - \beta \sin\gamma_0$$

Then the errors in the y and z components of the line of sight when system 1 is not level are given by

$$\left.\begin{aligned} \Delta L_{y1} &= \gamma \sin\gamma_0 \\ \Delta L_{z1} &= -\beta \sin\gamma_0 \end{aligned}\right\} \tag{B-5}$$

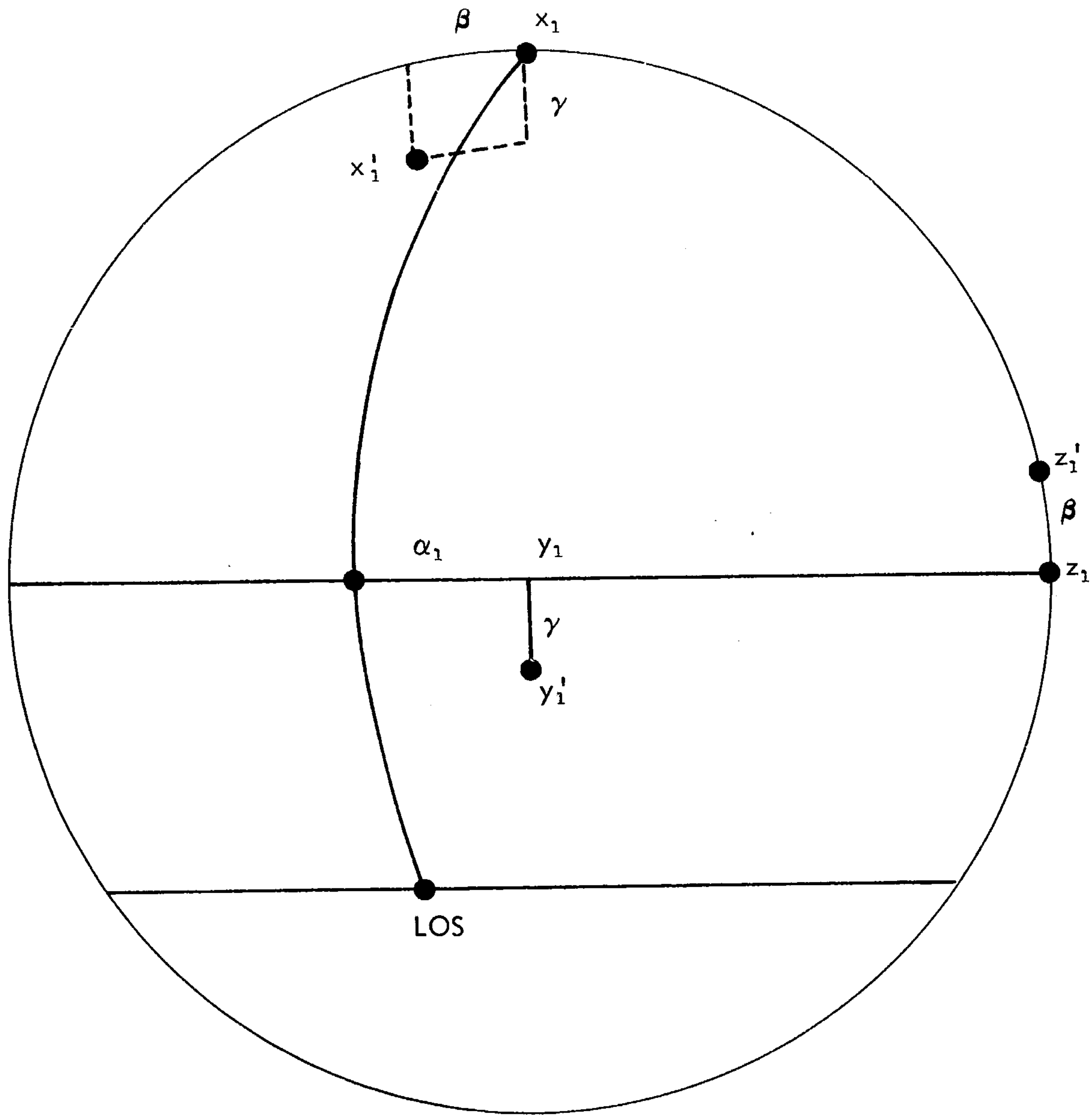


FIGURE B-2.

The error in α_1 is given by

$$\delta A = \frac{\partial \alpha 1}{\partial L_{y1}} \Delta L_{y1} + \frac{\partial \alpha 1}{\partial L_{z1}} \Delta L_{z1}$$

Using B-2

$$\delta A = \frac{1}{1 + \frac{(L_{y1})^2}{(L_{z1})^2}} \left\{ \left[\frac{1}{L_{z1}} \Delta L_{y1} \right] - \left[\frac{L_{y1}}{L^2_{z1}} \Delta L_{z_1} \right] \right\} \qquad \text{(B-6)}$$

From B-3

$$\begin{aligned} L_{y1} &= \cos \alpha_1 \cos \gamma_0 \\ L_{z1} &= -\sin \alpha_1 \cos \gamma_0 \end{aligned} \qquad \text{(B-7)}$$

Substituting B-5 and B-7 into B-6 yields

$$\delta A = \frac{1}{1 + \frac{\cos^2 \alpha 1}{\sin^2 \alpha 1}} \left\{ \frac{\gamma \sin \gamma_0}{\sin \alpha 1 \cos \gamma_0} \quad \frac{\beta \sin \gamma_0 \cos \alpha 1 \cos \gamma_0}{\sin^2 \alpha 1 \cos^2 \gamma_0} \right\}$$

$$\delta A = \tan \gamma_0 \left\{ - \gamma \sin \alpha_1 + \beta \cos \alpha_1 \right\} \qquad \text{(B-8)}$$

But $-\gamma \sin \alpha_1 + \beta \cos \alpha_1$ is the tilt about the horizontal projection of the line of sight and equals δ_L.

$\delta_A = \delta_L \tan \gamma_0$ which is the equation in Kennel's report.

An estimate of α_1 calculated using equation B-2 can be used in equation B-9 to determine a correction to be applied to the original estimate of α_1 to obtain a better estimate. The correction can be calculated using estimates for γ and β obtained from the erection system.

APPENDIX C
CLOSED FORM SOLUTIONS TO THE LEAST LEAST SQUARES EQUATION

The form of the linear least squares curve fit equation is

$$\begin{bmatrix} \hat{a}_0 \\ \\ \hat{a}_1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{K} i & \sum_{i=1}^{K} t_i \\ \sum_{i=1}^{K} t_i & \sum_{i=1}^{K} t^2{}_i \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=1}^{K} V_i \\ \sum_{i=1}^{K} t_i V_i \end{bmatrix} \qquad \text{(C-1)}$$

The estimate of the erection angle, θ is given by

$$\hat{\theta} = \frac{\hat{a}_1}{g}.$$

When the data is taken at constant time intervals δt

$$\left.\begin{aligned} &\sum_{i=1}^{K} = K \\ &\sum_{i=1}^{K} t_i = \frac{\delta t}{2} (K)(K+1) \\ &\sum_{i=1}^{K} t^2{}_i = \frac{\delta t^2}{6} (K)(K+1)(2K+1) \end{aligned}\right\} \qquad \text{(C-2)}$$

Using C-2, and C-1 yields

$$\hat{\theta} = \frac{\hat{a}_1}{g} = -\frac{6}{g\delta t(K)(K-1)} \sum_{i=1}^{K} V_i + \frac{12}{g\delta t^2 (K)(K-1)(K+1)} \sum_{i=1}^{K} tV_i \qquad \text{(C-3)}$$

Equation C-3 can also be used to derive an exact equation for the angular error resulting from a quadratic term in the data. The data is assumed to contain only a quadratic term of the form 1/2 $g\dot{\theta}t^2$. Using C-1, C-2 and the fact that

$$\sum_{i=1}^{K} t^3 = \frac{K^2 (K+1)^2}{4} \delta t^3$$

yields

$$\hat{\theta} = \frac{\hat{a}_1}{g} = (1/2)\,\dot{\theta}\; K\delta t \left(\frac{K+1}{K}\right) \qquad \text{(C-4)}$$

Equation C-4 is the exact equation for the error. In deriving the equations to be used to correct for earth's rate coupling in section III-F, it was assumed that the error was $\hat{\Theta}_\epsilon = 1/2\, \dot{\theta}\, T$ where $T = K\delta t$ (C-5)

Comparison of equations C-4 and C-5 shows that C-5 is correct to about 1% when 100 data points are taken at 1 second intervals. This error is perfectly acceptable.